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Another Cubic Associated with a Triangle

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Abstract. Let ABC be a triangle with side-lengths a, b, and c. For a point P in its plane, let AP_a, BP_b, and CP_c be the cevians through P. It was proved in [1] that the centroid, the Gergonne point, and the Nagel point are the only centers for which (the lengths of) BP_a, CP_b, and AP_c are linear forms in a, b, and c, i.e., for which $[AP_a\ BP_b\ CP_c] = [a\ b\ c]L$ for some matrix L. In this note, we investigate the locus of those centers for which BP_a, CP_b, and AP_c are quasi-linear in a, b, and c in the sense that they satisfy $[AP_a\ BP_b\ CP_c]M = [a\ b\ c]L$ for some matrices L and M. We also see that the analogous problem of finding those centers for which the angles $\angle BAP_a$, $\angle CBP_b$, and $\angle ACP_c$ are quasi-linear in the angles A, B, and C leads to what is known as the Balaton curve.

Key Words: triangle geometry, cevians, Nagel point, Gergonne point, irreducible cubic, Balaton curve, perimeter trisecting points, side-balanced triangle

MSC: 51M04, 51N35

1. Introduction

Let ABC be a non-degenerate triangle with side-lengths a, b, and c. For a point P in the plane of ABC, we let AP_a, BP_b, and CP_c be the cevians of ABC through P, and we define the *intercepts* x, y, and z *of* P by

$$x = BP_a, \quad y = CP_b, \quad z = AP_c \tag{1}$$

(see Fig. 1). Here, BP_a, CP_b and AP_c stand for directed lengths, where BP_a is positive or negative according as P_a and C lie on the same side or on opposite sides of B, and so on. To avoid infinite intercepts, we assume that P does not lie on any of the three exceptional lines passing through the vertices of ABC and parallel to the opposite sides.

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It is proved in [1, Theorem 1] that the only centers P for which x, y, and z are linear forms in a, b, and c are the centroid G, the Nagel point N, and the Gergonne point N'. This scarcity of centers defined by such linearity conditions is a result of the heavy restrictions that the cevians AP_a, BP_b, and CP_c are concurrent and that P is a *center function*, i.e., it assigns to each triangle a point in a manner which is symmetric with respect to permutations of the vertices.

Letting

$$x' = a - x, \quad y' = b - y, \quad z' = c - z, \tag{2}$$

we see that the condition for the concurrence of the cevians is given by Ceva's theorem as

$$xyz = x'y'z' = (a - x)(b - y)(c - z). \tag{3}$$

The restriction that P is a center function says that if $x = f(a, b, c)$, then $x' = f(a, c, b)$, and that y and z (respectively, y' and z') are obtained from x (respectively, x') by iterating the cyclic permutation $(a\ b\ c)$.

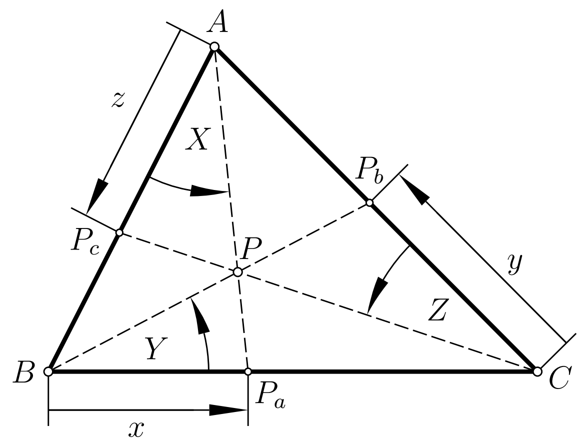

Figure 1: The triangle ABC with cevians and intercepts x, y, z

The linearity of x, y, and z in a, b, and c can be expressed as

$$[x\ y\ z] = [a\ b\ c]\, L, \tag{4}$$

where L is a 3×3 real matrix. One gets more centers by weakening this to take the form

$$[x\ y\ z]\, M = [a\ b\ c]\, L, \tag{5}$$

where M is not necessarily invertible. In view of (3), this can still be expected to result in unique solutions. Centers arising in this way are the subject of study in this note.

2. Triangle centers with quasi-linear intercepts

We start by proving that the matrix M can be assumed to be either invertible, in which case (5) is reduced to the linear case (4) already studied, or to the circulant matrix whose first column is $[1\ \ -1\ 0]^t$, where ^{t} denotes the transpose. In fact, if (x, y, z) satisfies an equation $\xi x + \eta y + \zeta z = f(a, b, c)$, then it must also satisfy $\zeta x + \xi y + \eta z = f(b, c, a)$. It follows that if the matrix M in (5) is to define a center, its column space must be invariant under the cycle

$(\xi, \eta, \zeta) \mapsto (\zeta, \xi, \eta)$. It is easy to see that the only such invariant subspaces in the $\xi\eta\zeta$-space $\mathbb{R}^3$ are the $\xi\eta\zeta$-space itself, the plane $\xi + \eta + \zeta = 0$ and the line $\xi = \eta = \zeta$. Thus M can be reduced to one of the following matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The first case is the one considered in (4) above and covered in [1, Theorem 1]. The last case results in infinitely many solutions that satisfy the cevian condition. Thus we may assume that M is the circulant matrix whose first column is $[1, -1, 0]^t$. The corresponding center function is thus defined by the requirement that $x - y$, and consequently $y - z$ and $z - x$, are linear forms in a, b, and c (together of course with the cevian condition).

A center function needs not be defined on the set $\mathbf{T}$ of all triangles, where $\mathbf{T}$ is identified with

$$\{(a, b, c) \in \mathbb{R}^3 : \ 0 < a < b + c, \ 0 < b < c + a, \ 0 < c < a + b\}.$$

However, it will be assumed that every center function is defined on a subset $\mathbf{U}$ of $\mathbf{T}$ having a non-empty interior. Since the interior of the zero set of a non-zero polynomial in any number of variables must be empty, it follows that a non-zero polynomial cannot vanish on $\mathbf{U}$ if $\mathbf{U}$ has a non-empty interior. This fact will be freely used.

Theorem 1 *Let $\mathcal{Z}$ be a center function and suppose that the intercepts x, y, and z of $\mathcal{Z}(ABC)$ are such that $x - y$ is a linear form in a, b, and c. Then there exists a unique $t \in \mathbb{R}$ such that*

$$\begin{bmatrix} x - y \\ y - z \\ z - x \end{bmatrix} = \begin{bmatrix} \frac{t+1}{2} & \frac{t-1}{2} & -t \\ -t & \frac{t+1}{2} & \frac{t-1}{2} \\ \frac{t-1}{2} & -t & \frac{t+1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{6}$$

Conversely, for every $t \in \mathbb{R}$ there is a center function $\mathcal{S}_t$ on some $\mathbf{U}_t \subseteq \mathbf{T}$ such that the intercepts x, y, and z of $\mathcal{S}_t(ABC)$ satisfy (6). The largest such $\mathbf{U}_t$ is defined by

$$(a, b, c) \in \mathbf{U}_t \iff ab + bc + ca \geq (a^2 + b^2 + c^2 - ab - bc - ca)t^2.$$

Consequently, $\mathcal{S}_t$ is defined on all triangles of $\mathbf{T}$ if and only if $-1 \leq t \leq 1$.

Proof: Let $\mathcal{Z}$ be a center function and suppose that the intercepts x, y, and z of $\mathcal{Z}(ABC)$ satisfy the equation

$$x - y = \xi a + \eta b + \zeta c \tag{7}$$

for some real numbers ξ, η, and ζ and for all triangles ABC in some $\mathbf{U}$. Adding (7) to its iterates $y - z = \zeta a + \xi b + \eta c$ and $z - x = \eta a + \zeta b + \xi c$, we see that

$$\xi + \eta + \zeta = 0. \tag{8}$$

The permutation $(A\ B)$ corresponds to the substitution

$$(x, y, z, a, b, c) \mapsto (y', x', z', b, a, c) = (b - y, a - x, c - z, b, a, c).$$

Applying this to (7), we obtain $(b - y) - (a - x) = \xi b + \eta a + \zeta c$ and hence $x - y = (\eta + 1)a + (\xi - 1)b + \zeta c$. From this and (7), we get $\xi - \eta = 1$. Using (8), we see that $\xi = (1 - \zeta)/2$ and $\eta = (-1 - \zeta)/2$. We obtain (6) by setting $\zeta = -t$.

Conversely, suppose $t \in \mathbb{R}$ is given, and let ABC be a triangle in $\mathbf{T}$ with side-lengths a, b, and c. Let

$$k = a^2 + b^2 + c^2 - ab - bc - ca, \quad p = ab + bc + ca, \quad d = (a-b)(b-c)(c-a). \tag{9}$$

First, we prove that if $p - kt^2 \geq 0$, then there exists a unique point V whose intercepts x, y, and z satisfy (6) and its iterates

$$y - z = \frac{t+1}{2}b + \frac{t-1}{2}c - ta\,, \quad z - x = \frac{t+1}{2}c + \frac{t-1}{2}a - tb\,. \tag{10}$$

Rewrite (6) and (10) as

$$2x - 2y = [t(b-c) + a] - [t(c-a) + b], \tag{11}$$
$$2y - 2z = [t(c-a) + b] - [t(a-b) + c], \tag{12}$$
$$2z - 2x = [t(a-b) + c] - [t(b-c) + a], \tag{13}$$

and let u be defined by

$$2x = t(b-c) + a + u, \tag{14}$$

where u is an indeterminate whose value is to be computed. Then similar expressions for $2y$, $2z$, $2x'$, $2y'$, and $2z'$ are found using (11), (12), and (13). It now remains to show that there is a unique u for which $xyz - x'y'z'$ vanishes. Denoting $4(xyz - x'y'z')$ by $g(u)$, we see that

$$\begin{aligned} 2g(u) &= (t(b-c)+a+u)(t(c-a)+b+u)(t(a-b)+c+u) \\ &\quad - (-t(b-c)+a-u)(-t(c-a)+b-u)(-t(a-b)+c-u) \end{aligned}$$

and therefore

$$g(u) = u^3 + (p - kt^2)u + (t^3 - t)d\,. \tag{15}$$

Then $g'(u) = 3u^2 + p - kt^2 \geq 0$. Therefore $g(u)$ is increasing and thus it has a unique zero, as desired.

Next, we prove that if $p - kt^2 < 0$, then there does not exist a unique point V whose intercepts x, y, z satisfy (6) and (10). Since a center function that is defined for ABC is expected to be defined for ACB, and since the quantity d given in (9) changes sign when ABC is replaced by ACB, we may assume that

$$t(t^2 - 1)d \leq 0. \tag{16}$$

Following the steps in the previous case, it is enough to show that the cubic polynomial $g(u)$ given in (15) has more than one zero. Equivalently, it is enough to prove that $g(u_0) \leq 0$, where $u_0 = \sqrt{(kt^2 - p)/3}$ is the larger zero of $g'(u)$. But

$$g(u_0) \;=\; u_0^3 - (kt^2 - p)u_0 + t(t^2-1)d \;=\; \frac{-2}{3}\left(kt^2 - p\right)^{3/2} + t(t^2-1)d \;<\; 0,$$

by (16), as desired.

It remains to prove the last statement. If $t^2 \leq 1$, then

$$\begin{aligned} p - kt^2 &\geq p - k \quad (\text{because } \; 2k = (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0) \\ &= 2(ab + bc + ca) - (a^2 + b^2 + c^2) \\ &= (a+b-c)(a+c-b) + (b+c-a)(b+a-c) + (c+a-b)(c+b-a) \\ &\geq 0 \quad (\text{by the triangle inequality}). \end{aligned}$$

Conversely, if $t^2 > 1$, then there exists $(a, b, c) \in \mathbf{T}$ for which $p - kt^2 < 0$, or equivalently $p/k < t^2$. In fact, $p/k = 1$ for the degenerate triple $(1, 1, 0)$ and $p/k = \infty$ for $(1, 1, 1)$ and therefore $p/k = t^2$ for some non-degenerate (a, b, c) of the form $a = b = 1$, $c > 0$. This completes the proof. □

In view of Theorem 1 above, one may, at times, restrict attention to the case $-1 \le t \le 1$. The next theorem is essentially a restatement of Theorem 1 under this restriction.

Theorem 2 *Let $\mathcal{Z}$ be a center function on $\mathbf{T}$ and suppose that the intercepts x, y, and z of $\mathcal{Z}(ABC)$ are such that $x - y$ is a linear form in a, b, and c. Then there exists a unique $t \in [-1, 1]$ such that (6) holds.*

Conversely, for every $t \in [-1, 1]$ there is a center function $\mathcal{Z}_t$ on $\mathbf{T}$ such that the intercepts x, y, and z of $\mathcal{Z}_t(ABC)$ satisfy (6). Furthermore, $\mathcal{Z}_t(ABC)$ lies inside ABC for all triangles.

Proof: We only need prove the last statement. Clearly, $\mathcal{Z}_t(ABC)$ lies inside ABC if and only if $0 < x < a$. In view of (14), this is equivalent to the requirement that $-a - t(b - c) < u < a - t(b - c)$, where u is the unique zero of the polynomial g given in (15). Now this would follow if we prove that $g(-a - t(b - c))$ and $g(a - t(b - c))$ have different signs. But

$$g(-a - t(b - c)) = aL_1L_2 \quad \text{and} \quad g(a - t(b - c)) = -aL_3L_4,$$

where

$$\begin{aligned} L_1 &= t(a + b - 2c) + (a + b), \qquad & L_2 &= t(a + c - 2b) - (a + c), \\ L_3 &= t(a + b - 2c) - (a + b), \qquad & L_4 &= t(a + c - 2b) + (a + c). \end{aligned}$$

Calculating each of the linear functions L_1, L_2, L_3, L_4 at $t = \pm 1$, we conclude that

$$L_1 > 0, \quad L_2 < 0, \quad L_3 < 0, \quad L_4 > 0$$

for all t in $[-1, 1]$. This completes the proof. □

Note 1: If $\mathcal{S}_t$ denotes the center function defined by (6), then it is easy to check that $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_{-1}$ are nothing but the centroid G, the Nagel center N and the Gergonne center N', respectively (Fig. 2). Before studying the locus of $\mathcal{S}_t(ABC)$ as t ranges in $[-1, 1]$, let us mention two other centers that naturally arise in connection with (6). Note first that (6) can be rewritten in the equivalent forms

$$x' + y = \frac{1 - t}{2}(a + b) + tc\,, \tag{17}$$

$$x + y' = \frac{1 + t}{2}(a + b) - tc\,. \tag{18}$$

The centers that satisfy

$$x' + y = y' + z = z' + x = \frac{a + b + c}{3}\,, \quad x + y' = y + z' = z + x' = \frac{a + b + c}{3}$$

correspond to $t = 1/3$ and $t = -1/3$, respectively. With reference to Fig. 1, they may be duly called *the (first and second) perimeter trisecting centers (or perimeter trisectors)*. The first one, $\mathcal{S}_{1/3}$, appears as Y_9 in [6, page 182] and as X_{369} in [7, page 267], where it is called the *trisected perimeter point*. The second one, $\mathcal{S}_{-1/3}$, does not seem to appear in the existing literature.

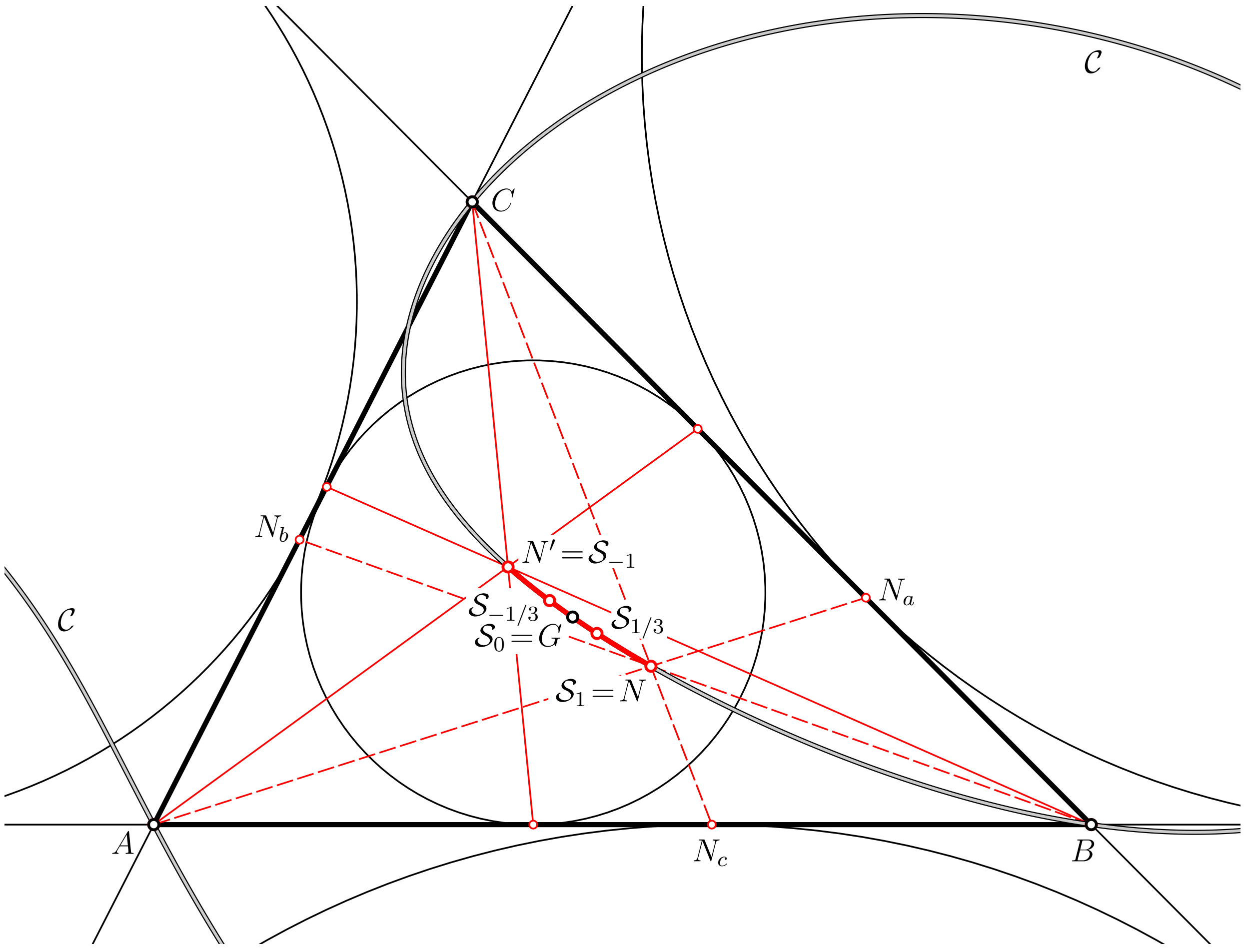

Figure 2: The red curve $\{\,\mathcal{S}_t \mid -1 \le t \le 1\}$ terminated by the Nagel point N and the Gergonne point N' together with its algebraic closure, the cubic $\mathcal{C}$

3. The locus of centers $\mathcal{S}_t$ and its algebraic closure

Theorems 3 and 4 give the trilinear equation of $\mathcal{S}_t$ as t ranges in $[-1, 1]$ and the trilinear coordinates of $\mathcal{S}_{1/3}$ and $\mathcal{S}_{-1/3}$. Note 2 compares the result with that of Yff as recorded in [7, page 267].

Theorem 3 *For $t \in [-1, 1]$, let $\mathcal{S}_t$ be the center defined by any of (6), (17), (18), or equivalently by $2x = t(b - c) + a + u$, where u is the solution of (15). Let the trilinear coordinates of $\mathcal{S}_t$ be denoted by (α, β, γ). Then the locus of $\mathcal{S}_t$ as t ranges in $[-1, 1]$ is a subset of the cubic curve $\mathcal{C}$ given by*

$$(1-2\cos A)\alpha(b^2\beta^2-c^2\gamma^2)+(1-2\cos B)\beta(c^2\gamma^2-a^2\alpha^2)+(1-2\cos C)\gamma(a^2\alpha^2-b^2\beta^2) = 0. \quad (19)$$

In the irreducible case $\mathcal{C}$ is the algebraic closure of $\{\,\mathcal{S}_t \mid -1 \le t \le 1\}$.

Proof: The barycentric coordinates $(x_a : x_b : x_c)$ and the trilinear coordinates $(\alpha : \beta : \gamma)$ of points in the plane of ABC are related by

$$(x_a : x_b : x_c) = (a\alpha : b\beta : c\gamma) \text{ and } \frac{x_c}{x_b} = \frac{x}{x'} \text{ etc.} \quad (20)$$

We conclude, e.g.,

$$(x_a : x_b : x_c) = (yz' : yz : y'z'). \tag{21}$$

After substituting (14) and the analogous expressions for x', y, y', z, z', we obtain a rational map $(t : u : 1) \mapsto (x_a : x_b : x_c)$ with

$$\begin{aligned} x_a &= (t(c-a)+b+u)\,(t(b-a)+c-u)\,, \\ x_b &= (t(c-a)+b+u)\,(t(a-b)+c+u)\,, \\ x_c &= (t(a-c)+b-u)\,(t(b-a)+c-u)\,. \end{aligned} \tag{22}$$

This is birational, because there is a rational inverse obeying

$$\begin{aligned} t &= \frac{c(x_b-x_a)(x_c+x_a)+b(x_b+x_a)(x_c-x_a)}{(2a-b-c)(x_b+x_a)(x_c+x_a)}\,, \\ u &= \frac{b(b-a)(x_b+x_a)(x_c-x_a)-c(c-a)(x_b-x_a)(x_c+x_a)}{(2a-b-c)(x_b+x_a)(x_c+x_a)}\,. \end{aligned} \tag{23}$$

We substitute these equations in (15) thus applying the birational map to the cubic curve $g(u) = 0$. After dividing by $4bc(2a-b-c)^2 x_a(x_c+x_a)(x_b+x_a)$ we obtain for the image curve $\mathcal{C}$ the barycentric equation

$$\mathcal{C}\colon\ k_a\, x_a(x_b^2 - x_c^2) + k_b\, x_b(x_c^2 - x_a^2) + k_c\, x_c(x_a^2 - x_b^2) = 0 \tag{24}$$

with coefficients

$$\begin{gathered} k_a = bc - b^2 - c^2 + a^2 = bc(1-2\cos A), \quad k_b = ac - a^2 - c^2 + b^2 = ac(1-2\cos B), \\ k_c = ab - a^2 - b^2 + c^2 = ab(1-2\cos C). \end{gathered} \tag{25}$$

This implies the trilinear equation (19). □

Note 2: In [7, Article 8.40, page 240], a cubic whose trilinear equation is of a form

$$x\alpha(\beta^2-\gamma^2) + y\beta(\gamma^2-\alpha^2) + z\gamma(\alpha^2-\beta^2) = 0$$

— similar to (19) — is denoted by $Z(P)$, where P is the point with trilinear coordinates $x : y : z$. These cubics are studied in detail in [8] and [2], and many examples of them have appeared in the literature.

In the sequel we list some properties of the cubic $\mathcal{C}$ obeying the barycentric equation (24):

1) $\mathcal{C}$ always passes through the points with barycentric coordinates $A_g = (-1 : 1 : 1)$, $B_g = (1 : -1 : 1)$, $C_g = (1 : 1 : -1)$ which beside the centroid $G = (1 : 1 : 1)$ are marked in Figs. 3–5 and connected by dashed lines. In these figures the red portion of $\mathcal{C}$ around the centroid G is the locus of $\mathcal{S}_t$ for $-1 < t < 1$ which is addressed in Theorem 2.

2) The cubic $\mathcal{C}$ is preserved under the mapping which exchanges the intercepts (x, x'), (y, y') as well as (z, z'). This is the well-known quadratic birational transformation of *isotomic points*, in barycentric coordinates

$$P = (x_a : x_b : x_c) \ \mapsto\ P' = (x_b x_c : x_a x_c : x_a x_b) = (1/x_a : 1/x_b : 1/x_c).$$

By (23) this is equivalent to changing the signs of t and u.

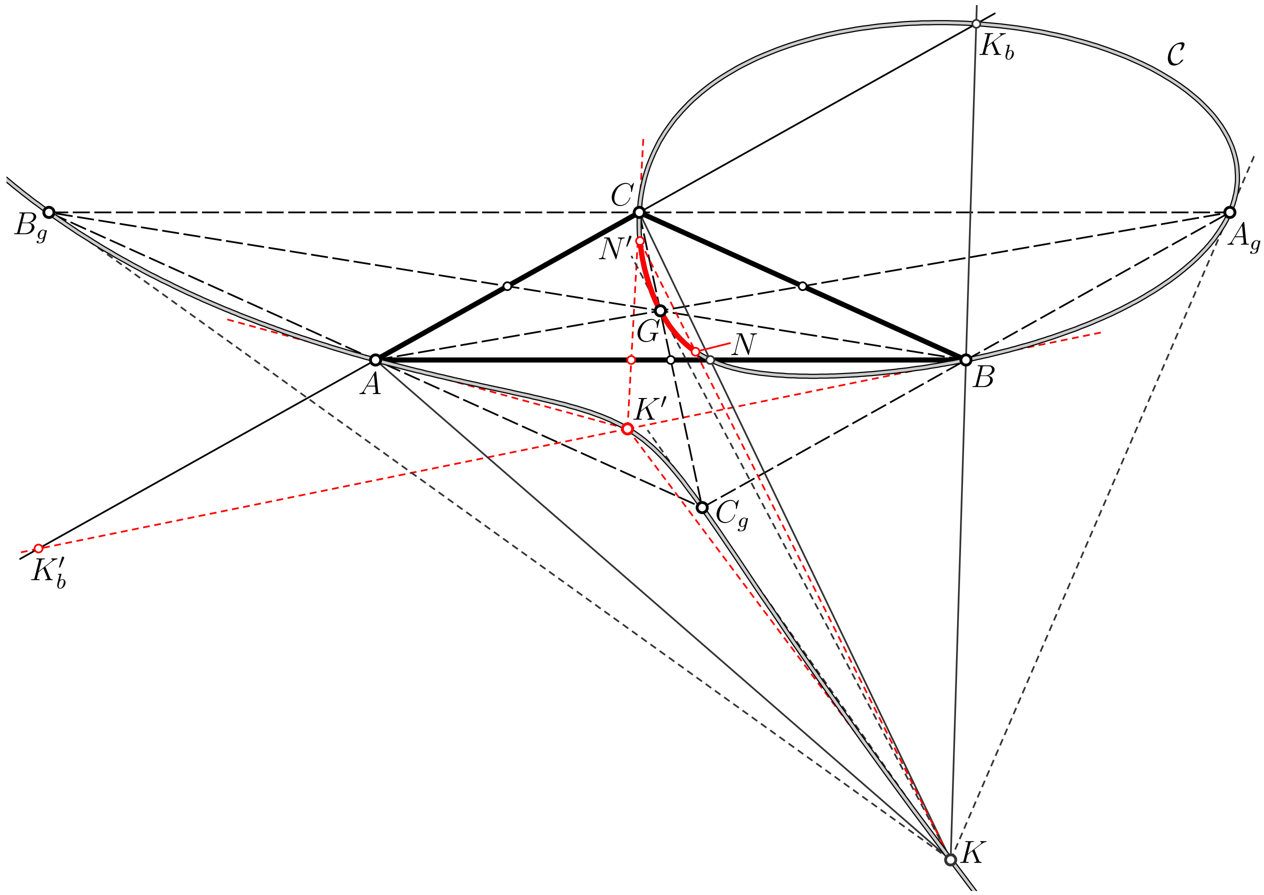

Figure 3: The irrational cubic (19)

3) The barycentric equation (24) of the cubic can also be written in the form

$$\det\begin{pmatrix} k_a & k_b & k_c \\ x_a & x_b & x_c \\ x_b x_c & x_a x_c & x_a x_b \end{pmatrix} = 0.$$

Hence, $\mathcal{C}$ is the algebraic closure of the set of points P which are aligned with their isotomic counterparts P' and the fixed point K with barycentric coordinates $(k_a : k_b : k_c)$ (Fig. 3). This implies:

- The lines connecting K with the vertices A, B, C pass also through the remaining points of intersection between $\mathcal{C}$ and the sides of the given triangle.
- The controid G and the points A_g, B_g and C_g remain fixed under the isotomic transformation. Therefore the tangent lines of $\mathcal{C}$ at these points pass through K.
- The isotomic transform of K is the point $K' = (k_b k_c : k_a k_c : k_a k_b)$. The tangent line at K passes through the corresponding K'.
- Point K is aligned with the Nagel point N and the Gergonne point N' (Fig. 3)[1] as well as with the centers $\mathcal{S}_{1/3}$ and $\mathcal{S}_{-1/3}$ (Fig. 2).

4) In the generic case the cubic $\mathcal{C}$ is irrational, i.e., it has no singularity. However, it is reducible in the following cases:

[1] Note that there is a pencil of cubics passing through the nine points A, B, C, G, A_g, B_g, C_g, N, and N'.

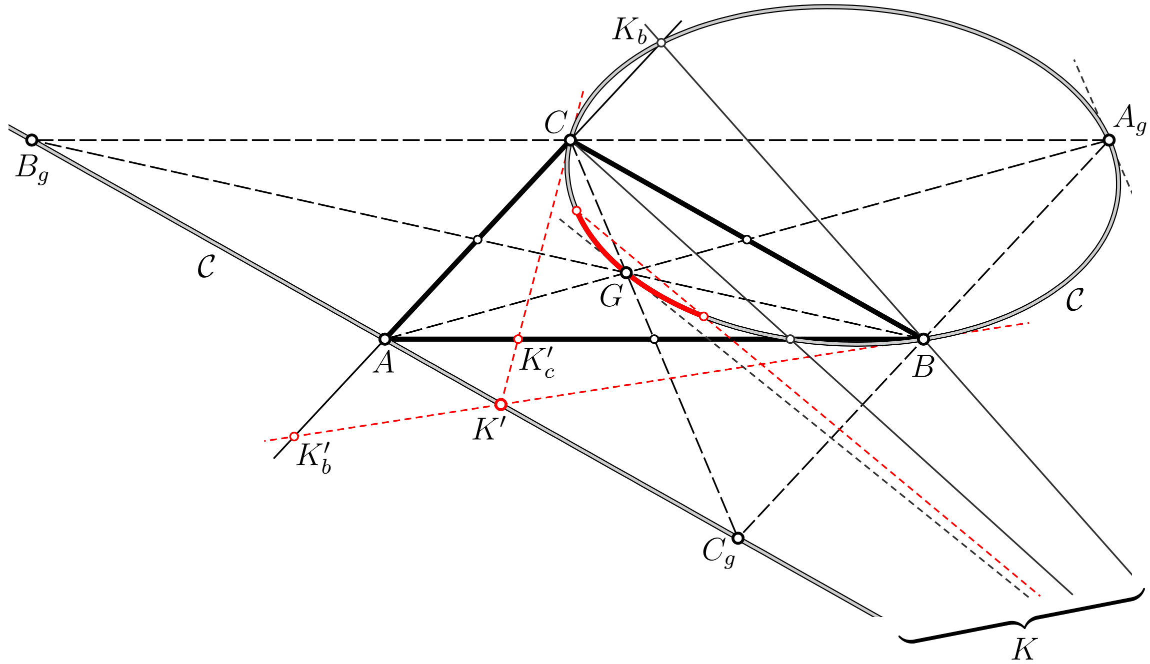

Figure 4: The cubic $\mathcal{C}$ splits for $b + c = 2a$

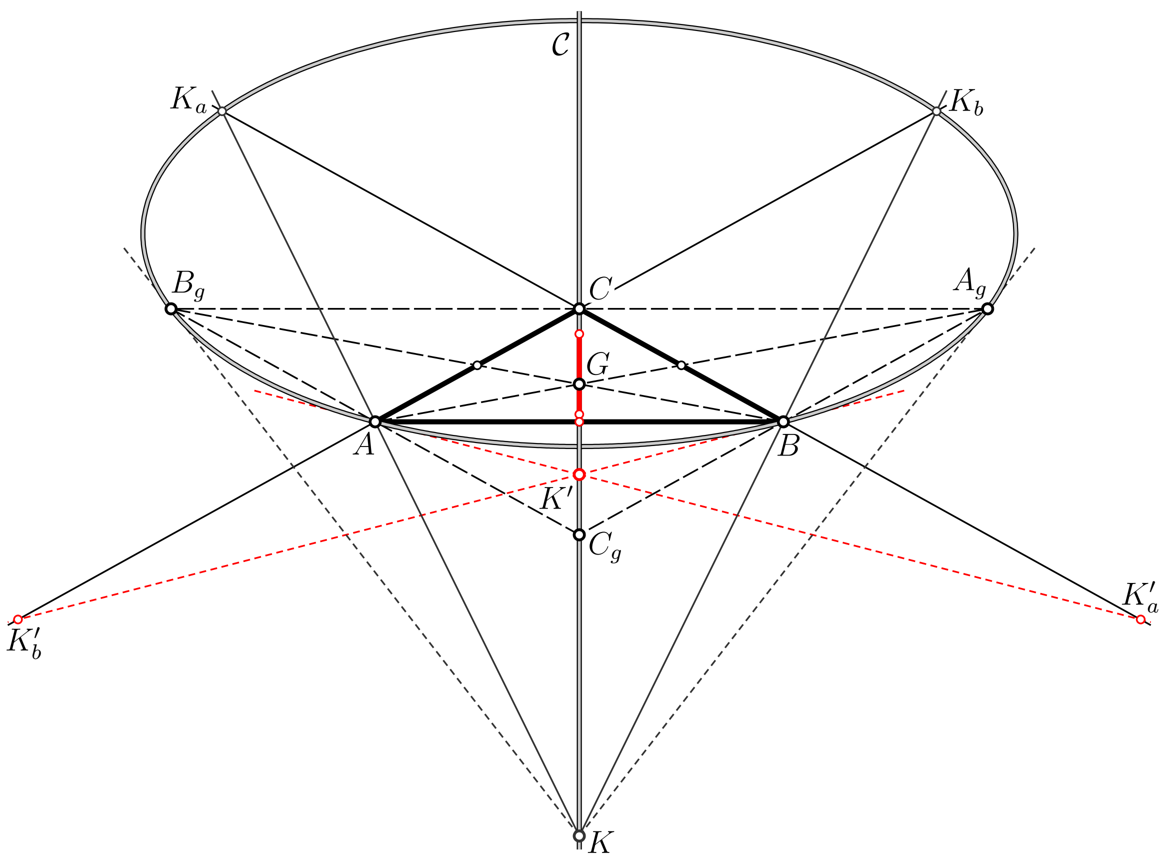

Figure 5: Isosceles triangle ($a = b$) with reducible cubic $\mathcal{C}$

- Under $a+b=2c$ it splits into a line and an ellipse (see Fig. 4). Then the coefficients k_a, k_b, k_c in the barycentric equation obey $k_a+k_b=0$, and the line $x_a+x_b=0$ is a component of $\mathcal{C}$. The same effect shows up under the permuted conditions $b+c=2a$ or $c+a=2b$. If in these cases the quadratic component of $\mathcal{C}$ is irreducible, then it is the algebraic closure of $\{\,\mathcal{S}_t \mid -1 \le t \le 1\}$.
- For an isosceles triangle ABC, e.g., with $a=b$ (see Fig. 5), two of the coefficients become equal. Then the cubic consists of the axis of symmetry and an ellipse. All centers $\mathcal{S}_t$ are located on the axis.
- For an equilateral triangle ABC the cubic splits into the three sides. The points $\mathcal{S}_t$ coincide with the center G.

Note 3: Triangles ABC with $b=c$ or $a=(b+c)/2$ were called *side-balanced* or *A-side-balanced* in [4]. Clearly, $a=(b+c)/2$ is equivalent to $\sin A=(\sin B+\sin C)/2$ and thus side-balanced means sine-balanced. The family of such triangles appeared in [4] in the following context: The Nagel point N of ABC has the circumcentral property $NB=NC$ iff ABC is side-balanced. This is equivalent to the following variation on Steiner-Lehmus theme [5]: If BN_b and CN_c are the cevians through the Nagel point N of ABC, then $NN_b=NN_c$ iff ABC is side-balanced.
The similar family of cosine-balanced triangles appeared in [4] in the form: The Nagel point N of ABC has the Fermat-Torricelli property $\angle AFB=\angle AFC$ iff ABC is cosine-balanced. The family of angle-balanced triangles (i.e., triangles in which $B=C$ or $A=(B+C)/2$ or equivalently $A=60°$) pops up very frequently in the literature.

5) The tangent lines of the cubic $\mathcal{C}$ at the vertices A, B, and C are given by the coefficients of x_a^2, x_b^2, and x_c^2 in the polynomial on the left hand side of (24). Hence they obey

$$k_c x_c - k_b x_b = 0, \qquad k_a x_a - k_c x_c = 0, \qquad k_b x_b - k_a x_a = 0,$$

respectively. These three lines meet at point K'.

6) Thus we know the quadruples of tangent lines passing through K and those through K'. If put into a particular order, their cross ratios are equal, and this is a projective invariant of the cubic $\mathcal{C}$. Computation gives

$$\delta = \frac{k_c^2-k_a^2}{k_c^2-k_b^2} = \frac{b(c-a)(2a+2c-b)(2b-a-c)}{a(c-b)(2b+2c-a)(2a-b-c)} \tag{26}$$

apart from permutations of (a,b,c). As long as neither the numerator nor the denominator vanishes, the cubic $\mathcal{C}$ is irreducible.
We summarize:

Theorem 4 *The cubic $\mathcal{C}$ (24), which in the irreducible case is the algebraic closure of the set $\{\,\mathcal{S}_t \mid -1 \le t \le 1\}$, has the following properties.*

1. *$\mathcal{C}$ is reducible for equilateral or isosceles triangles ABC or under one of the conditions $a+b=2c$, $b+c=2a$ or $c+a=2b$. Otherwise it is irrational with the characteristic cross ratio (26).*
2. *$\mathcal{C}$ is the algebraic closure of points P which are collinear with their isotomic transform P' and with the fixed point K with barycentric coordinates $(k_a : k_b : k_c)$ by (25).*

3. *$\mathcal{C}$ passes through the vertices A, B, C, through the centroid G and the points A_g, B_g, C_g with barycentric coordinates $(-1 : 1 : 1)$, $\ldots$, through the Nagel point N and the Gergonne point N', and through $K = (k_a : k_b : k_c)$ and its isotomic transform K'. The tangent lines of $\mathcal{C}$ at G, A_g, B_g, and C_g have the point K in common. The tangent lines at A, B, C, and K meet at K'.*

We conclude with the coordinates of the particular points $\mathcal{S}_{\pm 1/3}$:

Theorem 5 *The trilinear coordinates α, β, γ of the first perimeter trisecting center $\mathcal{S}_{1/3}$ are given by*

$$\gamma : \beta = c(c - b + 3a - U) : b(b - c + 3a + U),$$

where U is the unique real zero of

$$G(U) = U^3 + (9p - k)U - 8d,$$

and where k, p, and d are as given in (9).
The trilinear coordinates α, β, γ of the second perimeter trisecting center $\mathcal{S}_{-1/3}$ are given by

$$\gamma : \beta = b(c - b + 3a + W) : c(b - c + 3a - W),$$

where W is the unique real zero of

$$H(W) = W^3 + (9p - k)W + 8d.$$

Proof: The first statement follows by specifying $t = 1/3$ in (20) and (22) and letting $U = 3u$. Similarly for the second statement. □

Note 4: The trilinear coordinates of the first perimeter trisecting center $X_{369} = \mathcal{S}_{1/3}$ are given in [7, page 267] and [9] by the unsymmetric form

$$\alpha : \beta : \gamma = bc(v - c + a)(v - a + b) : ca(c + 2a - v)(a + 2b - v) : ab(v - c + a)(a + 2b - v),$$

where v is the unique real zero of

$$\begin{aligned} &2v^3 - 3(a + b + c)v^2 + (a^2 + b^2 + c^2 + 8ab + 8bc + 8ca)v \\ &-(b^2c + c^2a + a^2b + 5bc^2 + 5ca^2 + 5ab^2 + 9abc). \end{aligned}$$

Note 5: Considering the *angle analogue* of the above situation, and referring to Fig. 1, we define the angles X, Y, and Z associated with P by

$$X = \angle BAV, \quad Y = \angle CBV, \quad Z = \angle ACV$$

and we ask about those centers for which X, Y, and Z are linear or quasi-linear in the angles A, B, and C of triangle ABC. Here again, there are only three centers for which X, Y, and Z are linear in A, B, and C, namely, the incenter, the orthocenter, and the circumcenter; see [1, Theorem 2]. As for the quasi-linearity condition $[X\ Y\ Z]M = [A\ B\ C]L$, it is again equivalent to the requirement that $X - Y$ (and consequently $Y - Z$ and $Z - X$) is linear in A, B, and C. This in turn is clearly equivalent to the requirement that the central angles $\angle BPC$, $\angle CPA$, and $\angle APB$ are linear in A, B, and C. This follows from the observation $\angle BPC = A + Z + (B - Y)$. Thus the centers for which $X - Y$ is linear in A, B, and C are those centers for which the angles $\angle BPC$, $\angle CPA$, and $\angle APB$ are linear forms in A, B, and C. These centers are the subject of study in [3], and their locus is what was called the *Balaton curve*.

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