REMARKS ON BRICARD'S FLEXIBLE OCTAHEDRA OF TYPE 3

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ABSTRACT

This paper treats flexible polyhedra in the Euclidean 3-space. It is shown how the flexibility of Bricard's octahedra of Type 3 can be concluded with the aid of Ivory's theorem. Also other properties of this interesting flexible structure are presented.

Key Words: Flexibility, polyhedra, Bricard's octahedra, projective geometry.

1. INTRODUCTION

R. Bricard's flexible octahedra (Bricard (1897), compare Wunderlich (1965) or Stachel (1987)) play an essential role in the theory of flexible polyhedra. Most examples of continuously flexible polyhedra known until recent are somehow based on these octahedra (see e.g. Connelly (1978) or Dewdney (1992), note Stachel (2000)).

The first two types of Bricard's octahedra admit self-symmetries: In *Type 1* all pairs of opposite vertices are symmetric with respect to a common line, in *Type 2* two pairs are symmetric with respect to a common plane which passes through the remaining two vertices. We define Bricard's octahedra of *Type 3* by the property of being unsymmetric and nontrivially[1] flexible. Due to Bricard (1897) Type 3 admits two flat positions which can be determined in the following way (Fig. 1):

Let k_{AC}, k_{AB} be two different circles with the common center M, and let A_1, A_2 be two different points outside k_{AC} and k_{AB}. The tangent lines of k_{AB} passing through A_1 or A_2 define a quadrilateral. We specify (B_1, B_2) as a pair of opposite vertices.[2] Then $A_1B_1A_2B_2$ is a quadrangle with the four sides $A_1B_1, \ldots, B_2A_1$ tangent to k_{AB}. In the same way we specify a second quadrangle $A_1C_1A_2C_2$ tangent to the circle k_{AC}. Then (A_1, A_2), (B_1, B_2) and (C_1, C_2) are the pairs of opposite vertices of a flexible octahedron $\mathbf{O}$ in a flat position. The 8 faces of $\mathbf{O}$ are the triangles $A_iB_jC_k$ for any $i, j, k \in \{1, 2\}$.

We obtain a proper octahedron without self-symmetries under following assumptions:

(a) $B_1, \ldots, C_2$ are finite;

(b) A_1, A_2 are not aligned with M;

(c) the distances $\overline{A_1M}$ and $\overline{A_2M}$ are different.

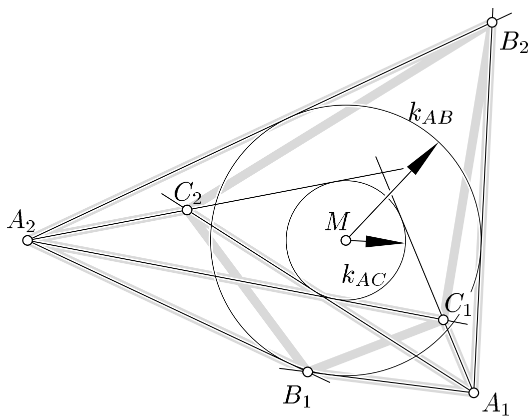

Figure 1: The flat position of Type 3

In the sequel a new proof for the flexibility of this octahedron $\mathbf{O}$ is given. The proof is based on standard results of Projective Geometry and on Ivory's theorem. It works also in the limiting case with one vertex at infinity which gives rise to a new flexible polyhedron with a prismatic part (see Section 5). The presented proof for the flexibility of Type 3 is – slightly modified – also valid in the elliptic and the hyperbolic 3-space. Some other properties of the particular hexagon $A_1 \ldots C_2$ of vertices of $\mathbf{O}$ in the flat position are presented in Section 4.

2. PROJECTIVE GEOMETRY OF THE CURVES OF SECOND CLASS

We recall a few results from Projective Geometry in the plane: The tangent lines of a conic c constitute a *regular curve $\widehat{c}$ of second class,* i.e., their coordinates[3] are the zeros of a regular quadratic function $\beta(\mathsf{u}, \mathsf{u})$, i.e., with

[1] At the trivially flexible cases two 'opposite' vertices are coinciding or two pairs of vertices are aligned.

[2] When k_{AB} happens to be tangent to the line A_1A_2 then the pair (B_1, B_2) is unique; one B-point is the point of contact.

[3] Let $\mathsf{x} = (x_0, x_1, x_2)$ be homogeneous *point coordinates* in the

a non-degenerate *polar form*[4] $\beta(\mathsf{u},\mathsf{v})$ for $\mathsf{u},\mathsf{v}\in\mathbb{R}^3$. A *singular* 2^{nd}-class curve is either the union $\widehat{p}_{PQ}$ of two pencils of lines at the points P, Q, resp., or a single pencil $\widehat{p}_R$ with multiplicity 2, a *"repeated pencil"*.

A pair of non-zero vectors u,v with $\beta(\mathsf{u},\mathsf{v})=0$ represents two lines which are *conjugate* with respect to $\widehat{c}$. For a given line $\mathsf{v}_0\mathbb{R}$ the set of all conjugate lines $\mathsf{u}\mathbb{R}$ is either a pencil of lines through the *pole*, or we have $\beta(\mathsf{u},\mathsf{v}_0)=0$ for all $\mathsf{u}\in\mathbb{R}^3$. In the latter case v_0 is included in the *radical* of the singular bilinear form β, i.e., the pole of $\mathsf{v}_0\mathbb{R}$ is indeterminate. In the case $\widehat{p}_{PQ}$ only the pole of the line PQ is indeterminate; for any other line l the pole is the fourth harmonic conjugate to the point of intersection $l\cap PQ$ with respect to P, Q. In the case $\widehat{p}_R$ for every line through R the pole is indeterminate while R is the pole of any other line.

For any polar form $\beta(\mathsf{u},\mathsf{v})$ there is a linear mapping $L\colon\mathbb{R}^3\to\mathbb{R}^3$ such that

$$\beta(\mathsf{u},\mathsf{v})=\mathsf{u}\cdot L(\mathsf{v}) \tag{1}$$

where the dot denotes the standard scalar product. Thus $L(\mathsf{v}_0)\mathbb{R}$ is the pole of the line $\mathsf{v}_0\mathbb{R}$, and the radical of β equals the kernel of L.

Any two different curves $\widehat{c}_1,\widehat{c}_2$ of 2^{nd} class span a one-parametric linear system, the *range* $\mathcal{R}:=[\widehat{c}_1\widehat{c}_2]$. The polar form of any $\widehat{c}\in\mathcal{R}$ reads

$$\beta(\mathsf{u},\mathsf{v})=\lambda_1\beta_1(\mathsf{u},\mathsf{v})+\lambda_2\beta_2(\mathsf{u},\mathsf{v}),\quad(\lambda_1,\lambda_2)\in\mathbb{R}^2,$$

provided, q_i denotes the polar form of $\widehat{c}_i$. In the generic case the range $\mathcal{R}$ consists of all 2^{nd}-class curves through the four common lines of $\widehat{c}_1$ and $\widehat{c}_2$. For a line $\mathsf{v}_0\mathbb{R}$ the poles with respect to all $\widehat{c}\in[\widehat{c}_1\widehat{c}_2]$ are represented by

$$(\lambda_1L_1(\mathsf{v}_0)+\lambda_2L_2(\mathsf{v}_0))\,\mathbb{R}. \tag{2}$$

A first example of a range is the set of circles with center M. This range includes the repeated pencil $\widehat{p}_M$ of diameters and the set $\widehat{p}_{I_1I_2}$ of isotropic lines, i.e., all lines passing through an *absolute point* I_1 or I_2. The homogeneous cartesian coordinates of these imaginary points read $(0:1:\pm i)$, $i^2=-1$.

As for all concentric circles the tangent line at I_j is the asymptote I_jM, this range is a special example of a *"contact range"* including all conics with two common line elements. The singular curves in such a contact range are
- the pencil $\widehat{p}_M$ of lines spanned by the common tangent lines (with multiplicity 2) and
- the set $\widehat{p}_{I_1I_2}$ of lines through the points of contact.

Another example of a range is the set of *confocal conics*. This range includes the singular curve $\widehat{p}_{F_1F_2}$ with the (real) focal points F_1, F_2 and again the set $\widehat{p}_{I_1I_2}$ of isotropic lines.

Any three second-class curves being not contained in a single range span a *two-parametric linear system* $\mathcal{S}$, which has the structure of a projective plane. This means: Any two different ranges $\mathcal{R}_1,\mathcal{R}_2\subset\mathcal{S}$ share exactly one 2^{nd}-class curve. For any two different curves $\widehat{c}_1,\widehat{c}_2\in\mathcal{S}$ the spanned range $[\widehat{c}_1\,\widehat{c}_2]$ is totally included in $\mathcal{S}$.

3. THE FLEXIBILITY OF TYPE 3

Lemma 1 *Let $A_1,\dots,C_2$ be the vertices of the flat position of an octahedron* **O** *of Type 3 — according to the previous construction (Fig. 1).*
Then the pairs of line pencils $\widehat{p}_{A_1A_2}$, $\widehat{p}_{B_1B_2}$ and $\widehat{p}_{C_1C_2}$ span a two-parametric linear system $\mathcal{S}$ which contains the repeated pencil $\widehat{p}_M$ and the set $\widehat{p}_{I_1I_2}$ of isotropic lines.

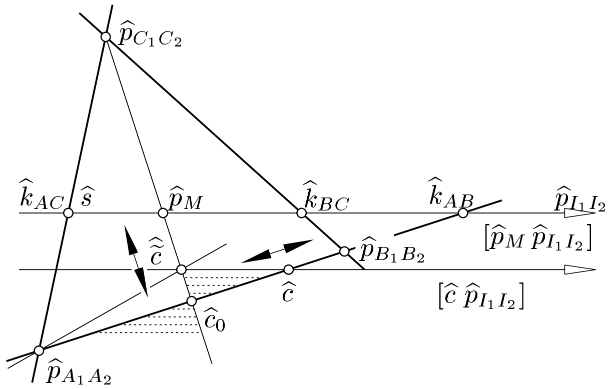

Figure 2: The two-parametric system $\mathcal{S}$ seen as a projective plane

Proof: Due to the construction displayed in Fig. 1 the range $[\widehat{p}_{A_1A_2}\,\widehat{p}_{B_1B_2}]\subset\mathcal{S}$ includes the circle $\widehat{k}_{AB}$. In the same way the range $[\widehat{p}_{A_1A_2}\,\widehat{p}_{C_1C_2}]\subset\mathcal{S}$ includes the circle $\widehat{k}_{AC}$. From $\widehat{k}_{AB},\widehat{k}_{AC}\in\mathcal{S}$ we conclude that the range $[\widehat{k}_{AB}\,\widehat{k}_{AC}]$ of concentric circles is a subset of $\mathcal{S}$. And this range contains the repeated pencil $\widehat{p}_M$ and the set $\widehat{p}_{I_1I_2}$ (compare Fig. 2).
$\mathcal{S}$ can also be spanned by $\widehat{p}_{A_1A_2}$ and the circles $\widehat{k}_{AB}$ and $\widehat{k}_{AC}$. Hence $\mathcal{S}$ doesn't depend on the ambiguous choice of the pairs (B_1,B_2) and (C_1,C_2). □

Consequences of Lemma 1 are:

Lemma 2 *(i) With any conic $\widehat{c}\in\mathcal{S}$ all conics confocal to $\widehat{c}$ are included in $\mathcal{S}$.*
(ii) Also the quadrangle $B_1C_1B_2C_2$ is tangent to a circle k_{BC} centered at M (see Fig. 4). This proves that no pair of vertices can be distinguished among (A_1,A_2), (B_1,B_2) and (C_1,C_2).

real projective plane. Then the line l given by the linear equation $u_0x_0+u_1x_1+u_2x_2=0$ has the homogeneous *line coordinates* $\mathsf{u}=(u_0,u_1,u_2)$. We write briefly $l=\mathsf{u}\mathbb{R}$.

[4] = symmetric bilinear form.

Proof: The second item can be concluded in the following way: The range $[\widehat{p}_{B_1B_2}\,\widehat{p}_{C_1C_2}]$ must intersect the range $[\widehat{p}_M\,\widehat{p}_{I_1I_2}]$ of concentric circles at a common curve $\widehat{s}$. It is easy to see (Fig. 2) that $\widehat{s}$ must be different from both, the pencil $\widehat{p}_M$ and the pair $\widehat{p}_{I_1I_2}$ of pencils of lines. Hence $\widehat{s}$ is a circle centered at M. □

Lemma 3 *For any conic c tangent to the sides of $A_1B_1A_2B_2$ there is a confocal conic $\widetilde{c}$ which passes through C_1 and C_2. All the conics $\widehat{\widetilde{c}}$ belong to a contact range as they have the common tangent line C_iM at C_i, $i = 1, 2$.*

Proof: For any conic c tangent to $A_1B_1A_2B_2$ its dual $\widehat{c}$ is included in $[\widehat{p}_{A_1A_2}\,\widehat{p}_{B_1B_2}] \subset \mathcal{S}$. Therefore the range $[\widehat{c}\,\widehat{p}_{I_1I_2}]$ of curves confocal to $\widehat{c}$ is contained in $\mathcal{S}$, too. There must be a curve $\widehat{\widetilde{c}}$ of intersection with the contact range $[\widehat{p}_M\,\widehat{p}_{C_1C_2}]$ of 2[nd]-class curves through the line elements (C_i, MC_i), $i = 1, 2$. Again we can exclude that the singular curves $\widehat{p}_M$ or $\widehat{p}_{C_1C_2}$ belong to $[\widehat{c}\,\widehat{p}_{I_1I_2}]$. Hence $\widehat{\widetilde{c}}$ has the properties stated in Lemma 3. □

Theorem 1 [Bricard (1897)] *The octahedron* **O** *with given flat position $A_1 \dots C_2$ is continuously flexible.*

Proof: We start with recalling focal properties of a one-sheet hyperboloid Φ with the equation

$$\Phi\colon\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \ \text{ for } a > b > 0,\ c > 0:$$

There is an affine transformation

$$\alpha_1\colon\ (x, y, z) \mapsto \left(\sqrt{a^2+c^2}/a,\ \sqrt{b^2+c^2}/b,\ 0\right)$$

which maps Φ into the plane $z = 0$ of symmetry. In particular each generator of Φ is mapped isometrically onto a tangent line of the *focal ellipse*

$$e\colon\ \frac{x^2}{a^2+c^2} + \frac{y^2}{b^2+c^2} - 1 = z = 0$$

of Φ. There is a second affine transformation

$$\alpha_2\colon\ (x, y, z) \mapsto \left(\sqrt{a^2-b^2}/a,\ 0,\ \sqrt{c^2+b^2}/c\right)$$

mapping Φ into $y = 0$ while the generators are transformed isometrically into tangent lines of the *focal hyperbola* [5]

$$h\colon\ \frac{x^2}{a^2-b^2} - \frac{z^2}{c^2+b^2} - 1 = y = 0.$$

e and h are confocal to the corresponding coplanar section of Φ. Note that tangent lines of the focal curves do never intersect the coplanar principal sections; e.g., the focal ellipse e lies in the exterior of Φ.

According to Ivory's theorem (see e.g. Stachel (2002)) for any two points $P, Q \in \Phi$ the following distances are equal:

$$\overline{P\,\alpha_i(Q)} = \overline{Q\,\alpha_i(P)}.$$

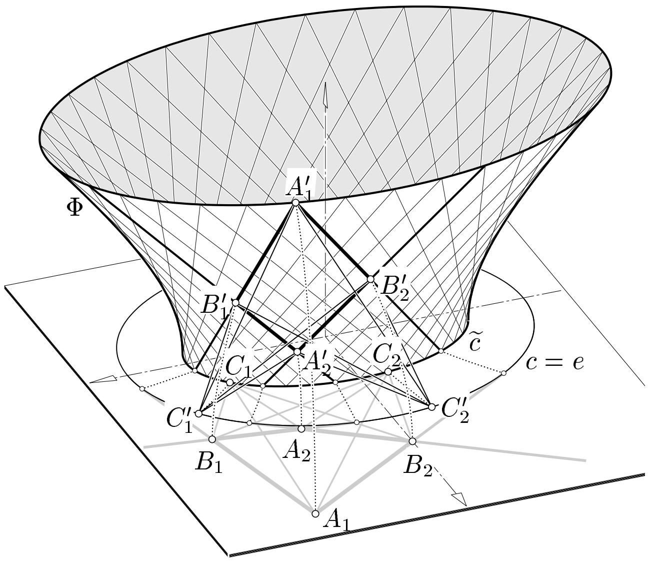

Figure 3: Proving the flexibility of Type 3 with Ivory's theorem

For proving the existence of a continuous set of octahedra being isometric to the flat position we note Lemma 3 which states: There is a continuous set of conics c tangent to $A_1B_1A_2B_2$ with a confocal $\widetilde{c}$ passing through C_1 and C_2. We see each $\widetilde{c}$ as a principal section of a one-sheet hyperboloid Φ and c as a coplanar focal curve (see Fig. 3).

The statement above reveals that there is a quadrangle $A_1'B_1'A_2'B_2'$ with sides on Φ which is mapped by the affine transformation α_1 or α_2 onto $A_1B_1A_2B_2$ while the lengths of all sides are preserved.[6] Under the same α_i the vertices $C_1, C_2 \in \widetilde{c}$ are mapped onto $C_1', C_2' \in c$, and Ivory's theorem concludes the proof that the spatial octahedron $A_1' \dots C_2'$ is isometric to the flat position.

However, two items remain to be proved:

- $\widetilde{c}$ needs to be inside the focal curve c, to say, no tangent line of c may intersect $\widetilde{c}$, and
- c and $\widetilde{c}$ must be of the same type.

For proving this, we note that there is a conic c_0 tangent to $A_1B_1A_2B_2$ and passing through both line elements (C_i, MC_i), $i = 1, 2$. Fig. 2 reveals

[5] The two affine transformations α_1, α_2 are limiting cases of a continuous set of affine transformations between Φ and any confocal one-sheet hyperboloid. These mappings are length-preserving for all generators of Φ. This is the basis for Henrici's flexible model of a one-sheet hyperboloid with linked rods representing the two sets of generators (see e.g. Hilbert (1996) or Stachel (1996)). The flat limiting positions of this model consist of tangent lines either of the focal ellipse e or the focal hyperbola h.

[6] $A_1'B_1'A_2'B_2' \in \Phi$ is unique only up to a reflection in the plane of the principal section $\widetilde{c}$.

$$\{\widehat{c}_0\} = [\widehat{p}_{A_1A_2}\,\widehat{p}_{B_1B_2}] \cap [\widehat{p}_M\ \widehat{p}_{C_1C_2}].$$

Now we start with $c = \widetilde{c} = c_0$ and use continuity arguments:

Let t denote any side of $A_1B_1A_2B_2$. While the 2[nd]-class curve $\widehat{\widetilde{c}}$ varies in the contact range $[\widehat{p}_M\ \widehat{p}_{C_1C_2}]$ the pole T of t with respect to $\widehat{\widetilde{c}}$ traces a line t' due to (2). For $\widehat{\widetilde{c}} = \widehat{c}_0$ we obtain T as the point T_0 of contact between t and c_0. For $\widehat{\widetilde{c}} = \widehat{p}_M$ the pole T coincides with M.

Now it depends on the choice of direction when starting from c_0: If T moves along t' torwards inside c_0, i.e., if T, T_0 are not separated by M and $S = t' \cap C_1C_2$, then the corresponding conic $\widetilde{c}$ will not intersect t. This results from properties of the polarity with respect to $\widetilde{c}$ and the involution of conjugate points on t'.

So these $\widetilde{c}$ are inside the confocal $c \in [\widehat{p}_{A_1A_2}\,\widehat{p}_{B_1B_2}]$. And in a neighborhood of c_0 neither the (affine) type of $\widetilde{c}$ nor that of c will change. It turns out that both conditions remain valid until T reaches the line C_1C_2 at S with $\widehat{\widetilde{c}} = \widehat{p}_{C_1C_2}$. This is the limiting position with the second flat position; the corresponding hyperboloid Φ degenerates into the focal conic of c_0. $\square$

4. PROPERTIES OF THE HEXAGON $A_1 \ldots C_2$

Let a_i, b_i, c_i denote the lines which connect M with A_i, B_i, C_i, resp., for $i = 1, 2$ (see Fig. 4).

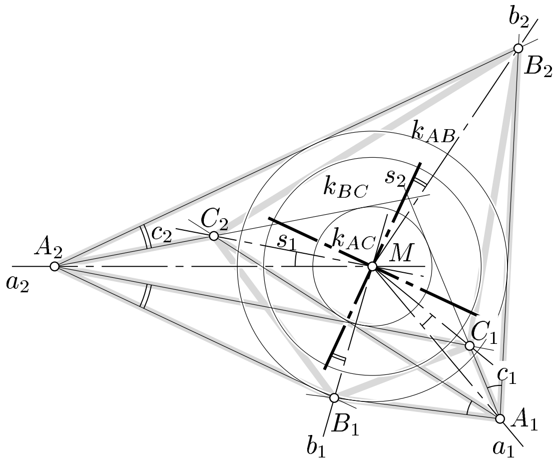

Figure 4: Symmetry properties of the flat position $A_1 \ldots C_2$

Lemma 4 *The pairs (a_1, a_2), (b_1, b_2) and (c_1, c_2) of lines through M have common axes s_1, s_2 of symmetry. At each of the six vertices (e.g. at A_i) the connecting lines with the other pairs (B_1, B_2 and C_1, C_2) are symmetric with respect to the line through M.*

Proof: The symmetry at M (see Fig. 4) is a consequence of Desargues' involution theorem: For the curves of the range $[\widehat{p}_{A_1A_2}\,\widehat{p}_{B_1B_2}]$ the tangent lines passing through M constitute an involution including the pair $MI_1 \mapsto MI_2$. Hence pairs of this involution are symmetric with respect to two perpendicular lines s_1, s_2. As all curves of a contact range through $\widehat{p}_M$ share the tangent lines through M, the involution includes the pairs of tangent lines for all curves in $\mathcal{S} \setminus \widehat{p}_M$.

The symmetry at the vertices is obvious because of the circles k_{AB}, k_{AC}, k_{BC}. $\square$

Lemma 5 *Let c_i' denote the line through C_i perpendicular to $c_i = MC_i$ (see Fig. 5). Then c_i and c_i' intersect the lines A_1A_2 and B_1B_2 at points being harmonic with respect to the incident pair of opposite vertices.*

Proof: $\mathcal{S}$ can be spanned by $\widehat{p}_M$, $\widehat{p}_{A_1A_2}$ and $\widehat{p}_{I_1I_2}$. Let $\beta_1, \beta_2, \beta_3$ be the corresponding polar forms and let L_1, L_2, L_3 be the linear mappings due to (1). Then for the line $l = \mathsf{v}\mathbb{R}$ the pole with respect to any $\widehat{c} \in \mathcal{S}$ is

$$(\lambda_1 L_1(\mathsf{v}) + \lambda_2 L_2(\mathsf{v}) + \lambda_3 L_3(\mathsf{v}))\,\mathbb{R}.$$

When l passes through M, we obtain $L_1(\mathsf{v}) = \mathsf{o}$. Therefore the poles trace a line l'. Due to the polarity with respect to $\widehat{p}_{I_1I_2}$ the line l' is perpendicular to l. On the other hand l' passes through the pole $\pi(l)$ (compare $\pi(c_1)$ in Fig. 5) of l with respect to $\widehat{p}_{A_1A_2}$. Without changing l' we can replace $\widehat{p}_{A_1A_2}$ by $\widehat{p}_{B_1B_2} \in \mathcal{S}$. $\square$

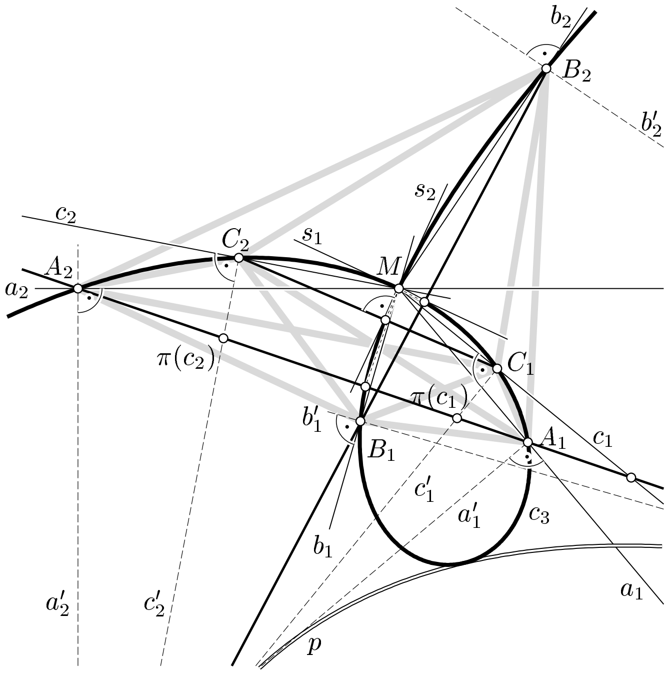

Figure 5: Strophoid c_3 as the locus of vertices

Theorem 2 *In the flat position of Type 3 the vertices $A_1, \ldots, C_2$ are located on a rational cubic c_3. This cubic has a node at M with perpendicular tangent lines*

s_1, s_2, and c_3 passes through the absolute points I_1, I_2. Therefore c_3 is a 'strophoidal' cubic with the following additional properties:

(i) c_3 is the locus of focal points of all conics included in $\mathcal{S}$.

(ii) c_3 is the locus of points of contact for tangent lines which can be drawn from M to any conic of S.

(iii) c_3 is the pedal curve of M with respect to the parabola p which is the envelope of the axes of symmetry for all non-circular conics included in $\mathcal{S}$.

(iv) The pairs of opposite vertices (A_1, A_2), (B_1, B_2) and (C_1, C_2) on c_3 are located on lines for which the pedal point of M is again a point of c_3.

Proof: We define c_3 als the set of intersection points $l \cap l'$ for any line l through M and the corresponding 'conjugate' l' with respect to all $\widehat{c} \in \mathcal{S}$. If $\widehat{c}$ is a conic with focal points F_1, F_2, then due to Lemma 2 also $\widehat{p}_{F_1 F_2}$ is included in $\mathcal{S}$. And for $l = MF_i$ the corresponding l' is the perpendicular line through F_i which implies $F_i \in c_3$.[7] The harmonic properties cited in Lemma 5 reveal that the lines l' connect corresponding points of a projectivity between A_1A_2 and the line at infinity. Hence the lines l' are tangent to a parabola p,[8] and $l \cap l'$ is the pedal point of M on the tangent line l' of p. From the projective generation of p we conclude that also A_1A_2 is tangent to p. Therefore the pedal point of M on A_1A_2 belongs to c_3, too. Finally, $\widehat{p}_{A_1 A_2}$ can be replaced by focal points of any other non-circular conic $\widehat{c} \in \mathcal{S}$. □

5. CONCLUDING REMARKS ON TYPE 3

The proof of the flexibility of Type 3 works in the same way in the projective models of the elliptic and the hyperbolic 3-space. The only difference is that $\widehat{p}_{I_1 I_2}$ has to be replaced by the dual of the absolute conic. So, also in these spaces there exist at least three types of flexible octahedra.

We can specify A_1, A_2 and M in Fig. 1 such that B_1 is a point at infinity. Ivory's theorem reveals also in this case a flexible polyhedron consisting of a pyramid and a prism. After reflecting the pyramid in a plane perpendicular to the prism we obtain a flexible polyhedron consisting of two pyramids and a cylindrical middle part.

A recently presented converse of Ivory's theorem (Stachel (2002)) allows to give a short new proof for the fact that Type 3 is the only flexible octahedron with a flat position — apart from trivial cases.

[7] In R. Bricard (1927) an approach to Type 3 is presented via properties of a "strophoidal" spatial cubic. This is exactly the analogon of c_3 in non-flat positions of the octahedron $\mathbf{O}$.

[8] p is the socalled Chalses' parabola associated to the pencil $\widehat{p}_M$ under the birational transformation of orthogonally conjugate lines with respect to any conic $\widehat{c} \in \mathcal{S}$ (compare Dingeldey (1903)).

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