Ivory's Theorem in the Minkowski Plane

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Abstract. According to the planar version of Ivory's theorem the net of confocal conics has the property that in each quadrangle formed by two pairs of conics the diagonals are of equal length. It turns out that this theorem is closely related to self-adjoint affine transformations. And from this point of view it is possible to prove the Minkowskian analogon of Ivory's Theorem in a more unified way for all six types of conics.

Key Words: Minkowkian geometry, pseudo-Euclidean geometry, confocal conics, Ivory's Theorem

MSC 2000: 51N25

1. Introduction

According to the planar Euclidean version of Ivory's Theorem the net of confocal conics has the property that in each quadrangle formed by two pairs of conics the two diagonals have the same length (see Fig. 1). Another formulation of this theorem uses the fact that for any two confocal conics k, k' of the same type an affine transformation α with $k \mapsto k'$ can be defined such that curves of the confocal net intersect k and k' orthogonally at corresponding points $X \in k$ and $X' = \alpha(X) \in k'$. Then Ivory's Theorem states

$$\overline{X_1\,\alpha(X_2)} = \overline{\alpha(X_1)\,X_2} \text{ for all } X_1, X_2 \in k.$$

This statement holds also for singular α when $k' = \alpha(k)$ degenerates into a set of points located on an axis of symmetry.

Ivory proved 1809 in [3] the 3D-version of this theorem by straight forward calculation using an appropriate parametrization (compare also [1, 2, 4, 5, 7]). Actually, this theorem holds in the Euclidean n-space for any $n > 1$ (see e.g. [6]). The aim of this paper is to demonstrate that Ivory's Theorem is also valid in the Minkowski plane $\mathbb{M}^2$ (pseudo-Euclidean plane). However, we avoid a straight forward computation separately for each of the six types of conics. Based on a lemma on self-adjoint affine transformations we give a more or less general proof in Section 3 by checking the system (17) of nonlinear equations.

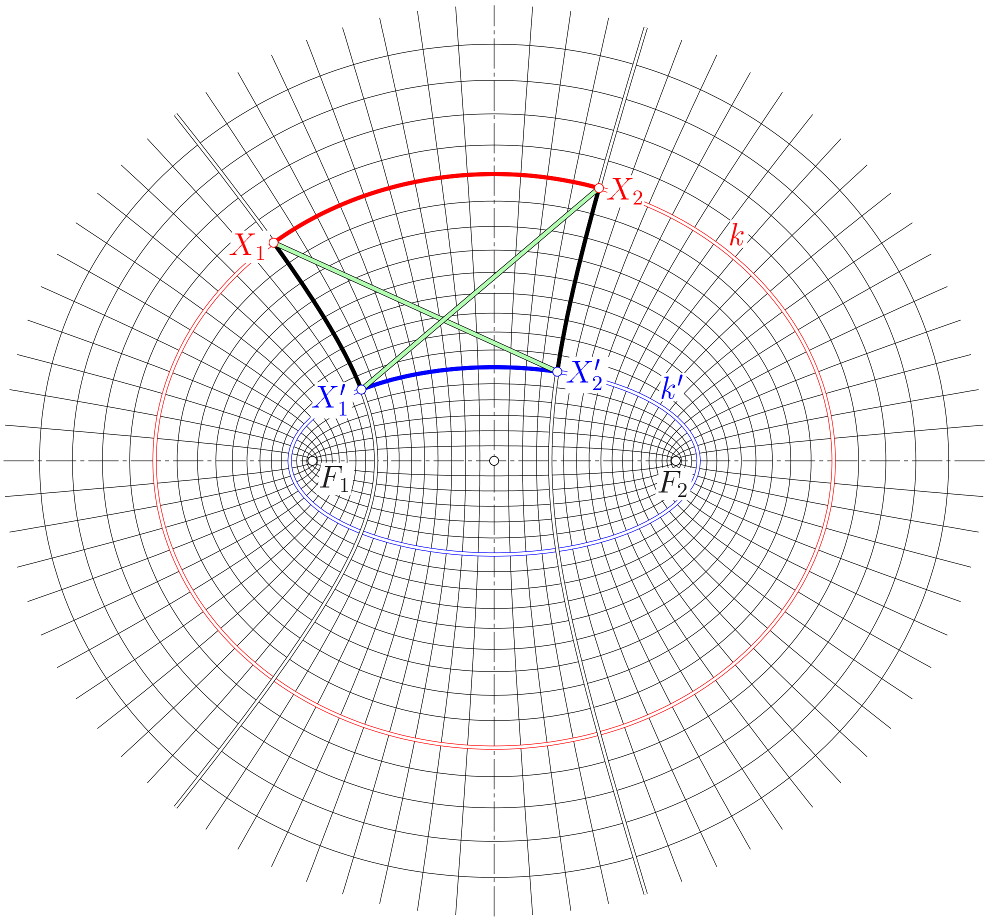

Figure 1: Ivory's Theorem in the Euclidean plane $\mathbb{E}^2$

The Minkowski plane $\mathbb{M}^2$ can be identified with the real affine plane where the underlying vector space $\mathbb{R}^2$ is endowed with a non-degenerate *indefinite* symmetric bilinear form ('scalar product'). The distance of points X, Y with coordinate vectors x, y is defined as

$$\overline{XY} = \|\mathsf{x} - \mathsf{y}\|_m := \sqrt{(\mathsf{x} - \mathsf{y})\cdot(\mathsf{x} - \mathsf{y})}.$$

This distance is either a nonnegative number or the product of a positive number and the imaginary unit i. A line segment XY as well as the spanned line $[XY]$ are called *lightlike (isotropic)*, *spacelike* or *timelike* if the length $\overline{XY}$ is zero, positive or imaginary, respectively.

We call an affine coordinate system *orthonormal* in the Minkowski sense when the 'scalar product' of two vectors $\mathsf{x} = (x_1, x_2)$, $\mathsf{y} = (y_1, y_2)$ can be written as

$$\mathsf{x}\cdot\mathsf{y} = x_1 y_1 - x_2 y_2. \tag{1}$$

Then the coordinate axes separate the two isotropic directions harmonically. In the sequel we only use orthonormal coordinates.

The figures in this paper are based on the standard model of $\mathbb{M}^2$ in $\mathbb{E}^2$ where the coordinate system is at the same time orthonormal in the Euclidean sense. Lightlike lines make an angle of 45° with the x_1-axis, spacelike lines have an inclination < 45°. In the case of ambiguities we use the prefix "*m*-" or "*e*-" at geometric terms in order to indicate whether they are meant in the Minkowskian or Euclidean sense, respectively.

Two conics are *confocal* in $\mathbb{M}^2$ if and only if their tangential equations span a linear system which contains the set of isotropic lines as a singular curve. This linear system usually contains also other singular curves — line pencils or pairs of line pencils. They can't be uniquely defined as point sets. However, they can show up in different ways as limiting curves of confocal conics.[1]

2. Types of conics in $\mathbb{M}^2$

Up to m-isometries and a commutation of the coordinate axes there are six types of conics to distinguish in $\mathbb{M}^2$. We present their equations in normal form:

A. *Circles:* They have the normal form

$$k\colon\ x_1^2 - x_2^2 = \sigma \text{ with } \sigma \neq 0. \tag{2}$$

We obtain the set of curves confocal to k by replacing the squared m-radius σ by a parameter $t \in \mathbb{R}$.

B. *Conics with two axes of symmetry:* Their equation in normal form reads

$$k\colon\ \frac{x_1^2}{\sigma} + \frac{x_2^2}{\tau} = 1 \text{ with } \sigma\tau(\sigma+\tau) \neq 0. \tag{3}$$

$e := \sqrt{\sigma+\tau}$ denotes the excentricity of these conics. Their m-focal points are

$$F_1 = (-e, 0),\ F_2 = (e, 0),\ F_3 = (0, -e),\ F_4 = (0, e)$$

(see Fig. 2). The set of conics confocal with k can be written as

$$\frac{x_1^2}{\sigma - t} + \frac{x_2^2}{\tau + t} = 1 \text{ for } t \in \mathbb{R} \setminus \{\sigma, \tau\}. \tag{4}$$

Under $\sigma, \tau > 0$ we get ellipses for $-\tau < t < \sigma$, hyperbolas for $t < -\tau$ or $t > \sigma$. The limiting curves of the ellipses for $t \to -\tau$ or $t \to \sigma$ are the closed line segments F_1F_2 or F_3F_4, respectively. The hyperbolas tend to pairs of aligned but disjoint half-lines terminated either by the focal points F_1, F_2 or by F_3, F_4.

C. *Hyperbolas with a spacelike and a timelike asympote:* These conics (see Fig. 4) have a center but no axis of symmetry. We use coordinate axes corresponding under the involution spanned by the isotropic directions and the asymptotes. Let $-1 < \sigma < 1$ denote the e-slope of the spacelike asymptote. Then we get the equation

$$\sigma(x_2^2 - x_1^2) + (1-\sigma^2)x_1x_2 = \tau \text{ with } \tau \neq 0. \tag{5}$$

The corresponding set of confocal conics reads

$$[\sigma + (1+\sigma^2)^2 t](x_2^2 - x_1^2) + (1-\sigma^2)x_1x_2 = \tau[1 + 8\sigma t - 4(1+\sigma^2)^2t^2],\ t \in \mathbb{R}. \tag{6}$$

In the Euclidean sense all these hyperbolas are orthogonal (see Fig. 4). The pairwise conjugate complex m-focal points are located on the e-isotropic lines $x_2 = \pm i x_1$.

[1]In $\mathbb{E}^2$ (see Fig. 1) the confocal ellipses with decreasing minor axes tend to the line segment terminated by the focal points F_1, F_2. The limiting curve of confocal hyperbolas with decreasing secondary axes consists of two half-lines terminated by F_1, F_2, respectively.

D. *Hyperbolas with one lightlike asymptote:* We specify the non-isotropic asymptote as x_1-axis and obtain

$$k\colon\ x_1x_2 - x_2^2 = \sigma \text{ with } \sigma \neq 0. \tag{7}$$

The confocal conics with equations

$$-tx_1^2 + x_1x_2 + (t-1)x_2^2 = \sigma(12t)^2 \text{ for } t \in \mathbb{R} \tag{8}$$

share the focal points $F_1 = (\sqrt{2\sigma}, \sqrt{2\sigma})$ and $F_2 = (-\sqrt{2\sigma}, -\sqrt{2\sigma})$.

E. *Parabolas with non-isotropic axis:* We choose the axis as x_1-axis and get the normal form

$$k\colon\ x_2^2 - 4\sigma x_1 = 0 \text{ with } \sigma \neq 0. \tag{9}$$

The confocal parabolas obeying

$$x_2^2 - 4(t+\sigma)(x_1 - t) = 0 \text{ for } t \in \mathbb{R} \tag{10}$$

share the focal point $F = (-\sigma, 0)$.

F. *Parabolas with a lightlike axis:* Their equation in normal form reads

$$k\colon\ (x_1 + x_2)^2 - 2\sigma(x_1 - x_2) = 0 \text{ with } \sigma \neq 0. \tag{11}$$

The confocal parabolas obey

$$(x_1 + x_2)^2 - 2\sigma(x_1 - x_2) + 2t(x_1 + x_2) + t^2 = 0 \text{ for } t \in \mathbb{R}. \tag{12}$$

3. Proof of Ivory's Theorem in $\mathbb{M}^2$

We follow the ideas presented in [6], Lemma 2, and stress the fact that Ivory's Theorem deals with pairs $(\mathsf{x}_j, \mathsf{x}'_j)$ of affinely related points[2] $\mathsf{x}_j \in k$, $\mathsf{x}'_j \in k'$ of two confocal conics k, k'. Are there curves k, k' with the 'Ivory property' $\|\mathsf{x}_1 - \mathsf{x}'_2\|_m = \|\mathsf{x}'_1 - \mathsf{x}_2\|_m$ at any affine transformation?

Let us start with *two* affine mappings:

$$\begin{aligned} \alpha\colon\ \mathbb{M}^2 \to \mathbb{M}^2, &\quad \mathsf{x} \mapsto \alpha(\mathsf{x}) = \mathsf{a} + l(\mathsf{x}),\\ \alpha^*\colon\ \mathbb{M}^2 \to \mathbb{M}^2, &\quad \mathsf{y} \mapsto \alpha^*(\mathsf{y}) = \mathsf{a}^* + l^*(\mathsf{y}) \end{aligned} \tag{13}$$

with $l, l^*\colon \mathbb{R}^2 \to \mathbb{R}^2$ denoting the induced linear mappings.

Suppose there are curves of Ivory type, i.e., point sets $\mathsf{X} = \{\mathsf{x}_1, \mathsf{x}_2, \dots\}$ and $\mathsf{Y} = \{\mathsf{y}_1, \mathsf{y}_2, \dots\}$ such that there are equal distances

$$\|\mathsf{x}_j - \alpha^*(\mathsf{y}_k)\|_m = \|\alpha(\mathsf{x}_j) - \mathsf{y}_k\|_m \text{ for all } \mathsf{x}_j \in \mathsf{X} \text{ and } \mathsf{y}_k \in \mathsf{Y}.$$

This gives rise to the equation $(\mathsf{x}_j - \alpha^*(\mathsf{y}_k))\cdot(\mathsf{x}_j - \alpha^*(\mathsf{y}_k)) = (\alpha(\mathsf{x}_j) - \mathsf{y}_k)\cdot(\alpha(\mathsf{x}_j) - \mathsf{y}_k)$, or after substitution of (13)

$$\mathsf{x}_j^2 - 2\mathsf{x}_j\cdot[\mathsf{a}^* + l^*(\mathsf{y}_k)] + [\mathsf{a}^* + l^*(\mathsf{y}_k)]^2 = [\mathsf{a} + l(\mathsf{x}_j)]^2 - 2[\mathsf{a} + l(\mathsf{x}_j)]\cdot\mathsf{y}_k + {\mathsf{y}_k}^2.$$

[2] From now on we identify points X with their coordinate vectors x.

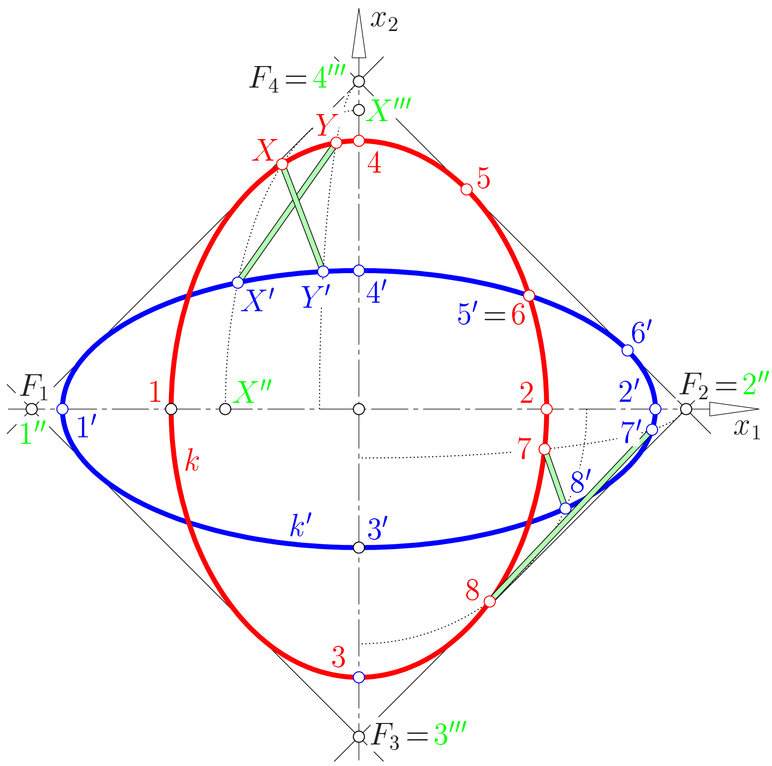

Figure 2: Confocal conics with two axes of symmetry (type B) in $\mathbb{M}^2$

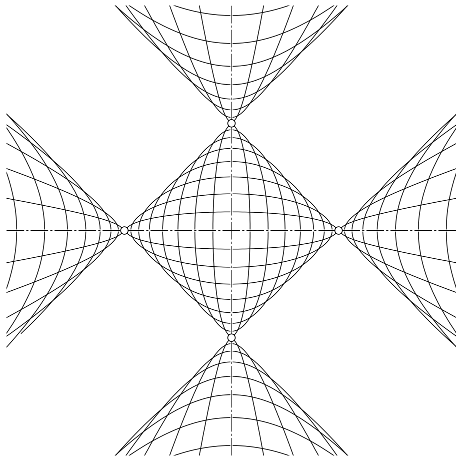

Figure 3: Net of confocal conics (type B) in $\mathbb{M}^2$

Now we specify that the linear mapping l^* is adjoint to l,[3] obeying

$$\mathsf{u}\cdot l^*(\mathsf{v}) = l(\mathsf{u})\cdot\mathsf{v} \ \text{ for all } \ \mathsf{u},\mathsf{v} \in \mathbb{R}^2. \tag{14}$$

Then in the equation above we can cancel the 'mixed' terms $\mathsf{x}_j\cdot l^*(\mathsf{y}_k) = l(\mathsf{x}_j)\cdot\mathsf{y}_k$ and separate the remaining terms such that those depending from x_j are placed on the left side, those depending from y_k on the right side:

$$\mathsf{x}_j^2 - l(\mathsf{x}_j)^2 - 2\mathsf{x}_j\cdot\mathsf{a}^* - 2l(\mathsf{x}_j)\cdot\mathsf{a} + \mathsf{a}^{*2} = {\mathsf{y}_k}^2 - l^*(\mathsf{y}_k)^2 - 2\mathsf{y}_k\cdot\mathsf{a} - 2\mathsf{a}^*\cdot l^*(\mathsf{y}_k) + \mathsf{a}^2.$$

As this equation shall hold for all $j,k \in \{1,2,\dots\}$, both sides must equal a constant $c \in \mathbb{R}$. This results in two quadratic functions

$$\begin{aligned} f(\mathsf{x}) &:= \ \mathsf{x}\cdot\mathsf{x} - l(\mathsf{x})\cdot l(\mathsf{x}) - 2\mathsf{x}\cdot[\mathsf{a}^* + l^*(\mathsf{a})] + \mathsf{a}^*\cdot\mathsf{a}^* - c, \\ g(\mathsf{y}) &:= \ \mathsf{y}\cdot\mathsf{y} - l^*(\mathsf{y})\cdot l^*(\mathsf{y}) - 2\mathsf{y}\cdot[\mathsf{a} + l(\mathsf{a}^*)] + \mathsf{a}\cdot\mathsf{a} - c \end{aligned} \tag{15}$$

with $f(\mathsf{x}_j) = g(\mathsf{y}_k) = 0$ for all $j,k \in \{1,2,\dots\}$. Hence X is the set of zeros of $f(\mathsf{x})$ and therefore a curve of second order. Conversely, for each constant $c \in \mathbb{R}$ all zeros x of f and y of g fulfil $\|\mathsf{x} - \alpha^*(\mathsf{y})\|_m = \|\alpha(\mathsf{x}) - \mathsf{y}\|_m$.

In the Ivory case the zero set X coincides with Y, and α equals α^*. Thus we have achieved the following

[3] It is proved in [6] that this condition is necessary if there are at least three non-collinear points $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3$. Here we only need the sufficiency.

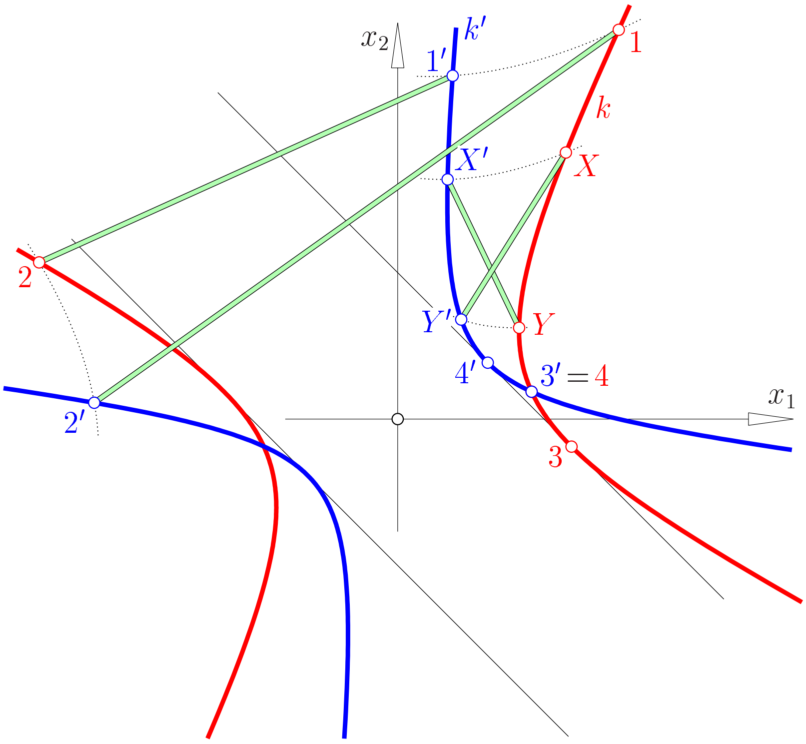

Figure 4: Confocal conics without any axis of symmetry (type C) in $\mathbb{M}^2$

Lemma 1: *For each self-adjoint affine transformation* α*, i.e. with* $l^* = l$ *and* $\mathsf{a}^* = \mathsf{a}$ *in* (13)*, the second-order curve* $\mathsf{X}: f(\mathsf{x}) = 0$ *according to* (15) *together with its image* $\alpha(\mathsf{X})$ *has the* Ivory *property* $\|\mathsf{x}_1 - \alpha(\mathsf{x}_2)\|_m = \|\alpha(\mathsf{x}_1) - \mathsf{x}_2\|_m$ *for any two zeros* $\mathsf{x}_1, \mathsf{x}_2$ *of* f*.*[4]

In order to prove Ivory's Theorem in $\mathbb{M}^2$, we associate to each conic k a self-adjoint affine transformation α such that k obeys the corresponding equation $f(\mathsf{x}) = 0$.

We use orthonormal coordinates and set up

$$\alpha = \alpha^* : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ with } a_{21} = -a_{12}.$$

The last equation is equivalent to the fact that l is self-adjoint with respect to the indefinite scalar product (1). Then the quadratic functions in (15) read explicitely

$$\begin{aligned} f(\mathsf{x}) = g(\mathsf{x}) = (1 - a_{11}^2 + a_{12}^2)x_1^2 - 2a_{12}(a_{11} + a_{22})x_1x_2 - (1 + a_{12}^2 - a_{22}^2)x_2^2 - \\ -2[(1 + a_{11})a_{10} + a_{12}a_{20}]x_1 + 2[(1 + a_{22})a_{20} - a_{12}a_{10}]x_2 + a_{10}^2 - a_{20}^2 - c\,. \end{aligned}$$

On the other hand, let a conic k be given by the equation

$$\gamma_{11}x_1^2 + 2\gamma_{12}x_1x_2 + \gamma_{22}x_2^2 + 2\gamma_{10}x_1 + 2\gamma_{20}x_2 + \gamma_{00} = 0 \qquad (16)$$

which of course is unique up to a factor $\lambda \in \mathbb{R} \setminus \{0\}$ only. The comparison of coefficients gives rise to the following system of equations for the unknown a_{jk}:

$$\begin{aligned} 1 - a_{11}^2 + a_{12}^2 &= \lambda\gamma_{11} & -(1 + a_{11})a_{10} - a_{12}a_{20} &= \lambda\gamma_{10} \\ -1 - a_{12}^2 + a_{22}^2 &= \lambda\gamma_{22} & -a_{12}a_{10} + (1 + a_{22})a_{20} &= \lambda\gamma_{20} \\ -a_{12}(a_{11} + a_{22}) &= \lambda\gamma_{12} & a_{10}^2 - a_{20}^2 - c &= \lambda\gamma_{00} \end{aligned} \qquad (17)$$

[4] Note that this holds for any symmetric bilinear 'scalar product' in any dimension (cf. [6]).

After the equations in the first column are solved for a_{11}, a_{12} and a_{22}, there are linear equations remaining for a_{10}, a_{20} and c. We deduce from the first column

$$a_{22}^2 - a_{11}^2 = \lambda(\gamma_{11} + \gamma_{22}) \quad \text{and} \quad \gamma_{12}(a_{22} - a_{11}) + (\gamma_{11} + \gamma_{22})a_{12} = 0. \tag{18}$$

Case 1, $\gamma_{12} = \gamma_{11} + \gamma_{22} = 0$; k is a circle (type A):
There are two solutions for α, either

$$a_{22} = a_{11} = \pm\sqrt{1 - \lambda\gamma_{11}},\ a_{12} = 0 \quad \text{or} \quad a_{22} = -a_{11},\ a_{12} = \pm\sqrt{\lambda\gamma_{11} + a_{11}^2 - 1} \quad \forall a_{11} \in \mathbb{R}.$$

In the normal form (2) of a circle k we have $\gamma_{11} = 1$, $\gamma_{10} = \gamma_{20} = 0$ and $\gamma_{00} = -\sigma$. Then in the first solution α is a dilatation with scaling factor $\sqrt{1-\lambda}$, $\lambda \le 1$, and $\alpha(k)$ is a concentric circle or a point ($\lambda = 1$). This is the Ivory case.

The second solution of α yields a bilinear mapping $l: \mathbb{R}^2 \to \mathbb{R}^2$ with the coordinate representation

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \sqrt{1-\lambda}\begin{pmatrix} \cosh\varphi & \sinh\varphi \\ -\sinh\varphi & -\cosh\varphi \end{pmatrix} \text{under } \lambda < 1, \text{ or } \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \text{under } \lambda = 1$$

for any $a \in \mathbb{R}$. In the regular case α is the product of a dilatation and an m-reflection in a diameter line; $\alpha(k)$ is again a circle concentric with k. The singular case is the only one which has no Euclidean counterpart: $\alpha(k)$ is an asymptote of k or ($a = 0$) it degenerates again into the center point of k.

Case 2, $\gamma_{12} = 0$, $\gamma_{11} + \gamma_{22} \neq 0$; k is of type B or E:
We solve (17) and (18) and obtain

$$a_{12} = 0, \quad a_{11} = \sqrt{1 - \lambda\gamma_{11}}, \quad a_{22} = \sqrt{1 + \lambda\gamma_{22}}.$$

This gives the 'classical' cases with Euclidean analoga:
For the normal form (3) of type B ($\gamma_{11} = 1/\sigma$, $\gamma_{22} = 2/\tau$) we obtain the affine transformation

$$\alpha\colon (x_1, x_2) \mapsto (x_1', x_2') = \left(x_1\sqrt{1 - \lambda/\sigma},\ x_2\sqrt{1 + \lambda/\tau}\right)$$

under $\lambda/\sigma \le 1$ and $\lambda/\tau \ge -1$. This is displayed in Fig. 2. The pairs of conics $(k, \alpha(k))$ within the confocal net (see Fig. 3) are any two ellipses or any two hyperbolas sharing their principal axis.

In the singular case $\lambda = -\tau$ with $\alpha\colon X \mapsto X$" (see Fig. 2) Ivory's Theorem reveals

$$\overline{XF_1} + \overline{XF_2} = \overline{X1"} + \overline{X2"} = \overline{X"1} + \overline{X"2} = \overline{12}.$$

Together with the second singular case $\lambda = \sigma$, $X \mapsto X'''$, we obtain e.g., that any ellipse k in $\mathbb{M}^2$ can be defined as

$$k = \left\{X \mid \overline{XF_1} + \overline{XF_2} = \overline{12} = C\right\} = \left\{X \,\middle|\, |\overline{XF_3} - \overline{XF_4}| = \overline{34} = Ci\right\} \text{ for a constant } C > 0.$$

In the parabolic case of type E ($\gamma_{11} = 0$, $\gamma_{22} = 1$, $\gamma_{10} = -2\sigma$) we get

$$\alpha\colon (x_1, x_2) \mapsto (x_1', x_2') = \left(\lambda\sigma + x_1,\ x_2\sqrt{1+\lambda}\right) \text{ for } \lambda \ge -1.$$

The parameters of corresponding parabolas k and $\alpha(k)$ have the same sign.

Case 3, $\gamma_{11}+\gamma_{22}=0$, $\gamma_{12}\neq 0$; k is of type C:
We obtain

$$a_{12}=\frac{-\lambda\gamma_{12}}{a_{11}+a_{22}},\ a_{22}=a_{11}\ \text{ and }\ q(a_{11}^2)=0\ \text{ for }\ q(x):=4x^2-4(1-\lambda\gamma_{11})x-\lambda^2\gamma_{12}^2=0\,.$$

The quadratic function $q(x)$ has always a positive zero since the coefficient of x^2 is positive and $q(0)<0$.
In the normal form (5) of type C we have $\gamma_{22}=\sigma=-\gamma_{11}$ and $\gamma_{12}=\frac{1}{2}(1-\sigma^2)$. This implies

$$a_{12}=-\frac{\lambda(1-\sigma^2)}{4a_{11}}\quad\text{and}\quad a_{11}^2-(1+\lambda\sigma)=\frac{\lambda^2(1-\sigma^2)^2}{16a_{11}^2}\,.$$

The affine transformation

$$\alpha\colon\begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\begin{pmatrix}x_1'\\x_2'\end{pmatrix}=\begin{pmatrix}a_{11} & -\frac{\lambda(1-\sigma^2)}{4a_{11}}\\ \frac{\lambda(1-\sigma^2)}{4a_{11}} & a_{11}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

is the product of an e-rotation and a dilatation[5] and maps k onto a confocal conic obeying (6) with $\lambda=4t$ (compare Fig. 4).

Case 4, $\gamma_{11}+\gamma_{22}\neq 0$ and $\gamma_{12}\neq 0$; k is of type D or F:
We deduce from (18)

$$a_{12}=\frac{\gamma_{12}(a_{11}-a_{22})}{\gamma_{11}+\gamma_{22}}$$

and substitute this in the first equation of (17). Replacing a_{22}^2 from (18) results in

$$a_{22}=\frac{(2\gamma_{12}^2-\gamma_{11}^2-2\gamma_{11}\gamma_{22}-\gamma_{22}^2)a_{11}^2-(\gamma_{11}+\gamma_{22})\left[(\gamma_{11}+\gamma_{22})(\lambda\gamma_{11}-1)-\lambda\gamma_{12}^2\right]}{2\gamma_{12}^2a_{11}}.$$

Then (18) gives rise to a biquadratic equation for a_{11}. However, for the types D with equation (7) ($\gamma_{11}=0$, $\gamma_{12}=\frac{1}{2}$, $\gamma_{22}=-1$) and F with equation (11) ($\gamma_{11}=\gamma_{12}=\gamma_{22}=1$, $\gamma_{10}=-\gamma_{20}=-\sigma$,) this equation is quadratic only, and it has always real solutions.

The affine transformation α for type D reads

$$\alpha\colon\begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\begin{pmatrix}x_1'\\x_2'\end{pmatrix}=\frac{1}{\mp 2\sqrt{4-2\lambda}}\begin{pmatrix}\lambda-4 & \lambda\\ -\lambda & 3\lambda-4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},\quad \lambda<2,$$

and the image $\alpha(k)$ obeys (8) with $\lambda=4t$.

For type F we get $a_{11}=\pm(\lambda-2)/2$. It turns out that the linear equations for a_{10} and a_{20} in (18), right column, are solvable only with the lower sign of a_{11}. We then obtain for type F the affine transformation

$$\alpha\colon\begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\begin{pmatrix}x_1'\\x_2'\end{pmatrix}=\frac{\lambda\sigma}{4}\begin{pmatrix}2+\lambda\\2-\lambda\end{pmatrix}+\frac{1}{2}\begin{pmatrix}2-\lambda & -\lambda\\ \lambda & 2+\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

[5] α is an e-similarity. This can also be concluded from the fact that the singular curves in the confocal net, the e-isotropic lines, intersect k and $\alpha(k)$ at corresponding points. Hence these e-isotropic lines remain fixed under α.

The equation of $\alpha(k)$ coincides with (12) under $t = -2\lambda\sigma$.

Thus we have proved

Theorem: *1. For any conic k in the Minkowski plane $\mathbb{M}^2$ there is a self-adjoint affine transformation α such that k obeys the corresponding equation $f(\mathsf{x}) = 0$ according to* (15).
2. Ivory*'s Theorem is true in $\mathbb{M}^2$ for all six types of conics.*

It can be verified that in all cases the path $\alpha(\mathsf{x}_0)$ of any point $\mathsf{x}_0 \in k$ for variable λ is again located on a curve of the confocal net. A general proof for this is left for a future publication — as well as a proof for the fact that any regular self-adjoint α maps $k\colon f(\mathsf{x}) = 0$ (given by (15)) onto a confocal conic $\alpha(k)$.

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