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On the Flexibility and Symmetry of Overconstrained Mechanisms

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In kinematics a framework is called overconstrained if its continuous flexibility is caused by particular dimensions; in the generic case a framework of this type is rigid. Famous examples of overconstrained structures are the Bricard octahedra, the Bennett isogram, the Grünbaum framework, Bottema's 16-bar-mechanism, Chasles' body-bar framework, Burmester's focal mechanism or flexible quad meshes. The aim of this paper is to present some examples in detail and to focus on their symmetry properties. It turns out that only for a few a global symmetry is a necessary condition for flexibility. Sometimes there is a hidden symmetry, and in some cases, e.g., at the flexible type-3 octahedra or at discrete Voss-surfaces, there is only a local symmetry. However, there remain overconstrained frameworks where the underlying algebraic conditions for flexibility have no relation to symmetry at all.

Key words: overconstrained mechanism, Grünbaum framework, Kokotsakis mesh, flexible bipartite framework, Burmester's focal mechanism.

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1. Introduction

Let $\mathcal{F} = (V, E)$ be a *bar-and-joint framework* in the d-dimensional Euclidean space $\mathbb{E}^d$ with the vertex set

$$V = \{\mathbf{x}_1, \dots, \mathbf{x}_v\},\ \mathbf{x}_i \in \mathbb{R}^d \ \text{ for all } \ i \in I = \{1, \dots, v\}$$

and the edge set

$$E \subset \{(i, j) \mid i < j,\ (i, j) \in I\}\,.$$

We denote the edge lengths by

$$l_{ij} := \|\mathbf{x}_i - \mathbf{x}_j\| \ \text{ for all } \ (i, j) \in E.$$

We call each v-tupel $(\mathbf{x}'_1, \dots, \mathbf{x}'_v) \in \mathbb{R}^{vd}$ with the same edge lengths l_{ij} for all $(i, j) \in E$ a *realization* of $\mathcal{F}$, and of course we are interested in mutually incongruent realizations.

$\mathcal{F}$ is called *continuously flexible* or — by short — *flexible*, if its spatial form can be changed analytically with respect to k parameters, $k > 0$, while its edge lengths remain unaltered. The maximum k is called *degree of freedom* (d.o.f.). The different realizations are also called *flexions*, and the continuous movement of $\mathcal{F}$ is called a *self-motion*.

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In the generic case a parameter count reveals that for a framework with v vertices and e edges in $\mathbb{E}^d$ we obtain

$$k = vd - e - \frac{d(d+1)}{2} \tag{1.1}$$

as the degree of freedom. A framework is called *overconstrained* if it is flexible due to its particular edge lengths though in the generic case its d.o.f. would be ≤ 0. On the other hand, in particular cases the degree of freedom of a framework can be 0 though the equation above gives $k \geq 1$; as an example look at a four-bar in $\mathbb{E}^2$, i.e., $v = e = 4$, where one edge length equals the sum of the three other lengths.

In the sequel we focus on overconstrained frameworks. We present a personal selection of different examples and explain in a more or less unified mathematical way why they are flexible. Among a few of them the symmetry of the framework is crucial for its flexibility, and the self-motion is symmetry preserving. In general, continuous flexibility is of algebraic nature, and the challenge is to find the geometric meaning of the algebraic conditions.

2. Flexible bipartite frameworks and Ivory's theorem

A framework is called *bipartite* if its edge graph is bipartite, i.e., its vertices (or *knots*) can be subdivided into two classes $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ such that the edges (or *bars*) always connect vertices from different classes. It turns out that the flexibility of bipartite frameworks is always related to Ivory's theorem [1].

To recall, the two-dimensional version of Ivory's theorem states that in the orthogonal net of confocal conics each curvilinear quadrangle has diagonals of equal length (Fig. 1a). In other words: If α is an affine transformation mapping the conic k onto a confocal conic k' while the axes of symmetry are kept fixed, then

$$\|\alpha(\mathbf{x}) - \mathbf{y}\| = \|\mathbf{x} - \alpha(\mathbf{y})\| \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in k.$$

Nomenclature

$\mathbb{E}^d$	d-dimensional Euclidean space
$\mathbb{S}^2$	sphere in $\mathbb{E}^3$
$A_i, B_j, V_k, \dots$	points, vertices of a framework
$\mathbf{a}_i, \mathbf{b}_j, \mathbf{x}_k, \dots$	position vectors of points or vertices, but also points
$\|\mathbf{v}\|$	norm of the vector $\mathbf{v}$
$\varepsilon_i, \varphi_k, \dots$	planes
$\boldsymbol{f}_0, \boldsymbol{f}_1, \dots$	faces (facets) of a polyhedron or quad mesh
$\mathrm{L}_i, \mathrm{R}_j$	tetrahedra
$\alpha, \beta, \gamma, \varphi_i, \psi, \dots$	angle measures
$\Sigma, \Sigma', \Sigma_1, \Sigma_2, \dots$	systems (= rigid bodies) involved at a mechanism
Σ_j/Σ_k	motion of system Σ_j w.r.t. system Σ_k
I_{jk}	instant axis of motion Σ_j/Σ_k

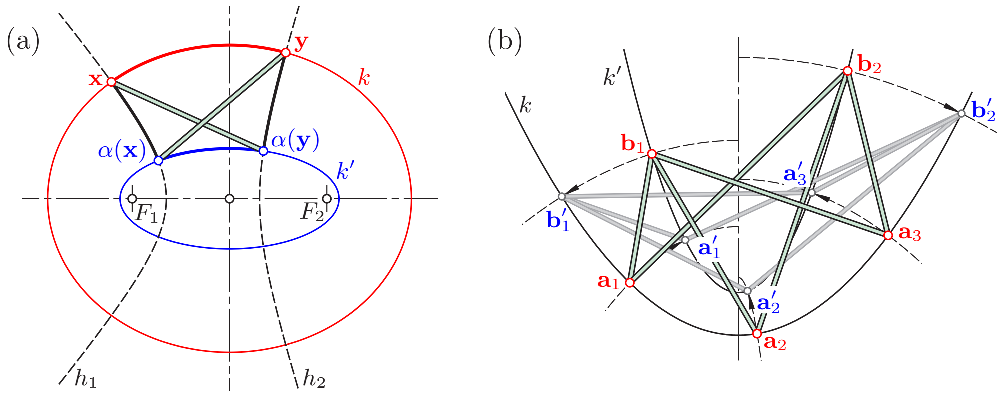

Figure 1. Ivory's theorem and two incongruent realizations of a bipartite framework

This is also valid for singular α. It turns out that corresponding points $\mathbf{x}$ and $\alpha(\mathbf{x})$ are always located on the same second conic of the confocal set (note the hyperbolas h_1, h_2 in Fig. 1a).

In Fig. 1b it is shown how by Ivory's theorem a planar bipartite framework with $\mathbf{a}_i \in k$ and $\mathbf{b}_j \in k'$ can be transformed into an incongruent realization with $\mathbf{a}'_i \in k'$ and $\mathbf{b}'_j \in k$.

Also the converse statement is true — and even in each dimension [2]: For any two incongruent realizations in $\mathbb{E}^d$ there is a displacement of one of them such that finally the two realizations are in "Ivory position", i.e., the vertices $\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{b}'_1, \dots, \mathbf{b}'_n$ and $\mathbf{a}'_1, \dots, \mathbf{a}'_m, \mathbf{b}_1, \dots, \mathbf{b}_n$ are placed on two confocal quadrics of the same type, respectively, and the pairs $\mathbf{a}_i \mapsto \mathbf{a}'_i$ and $\mathbf{b}'_j \mapsto \mathbf{b}_j$ are corresponding under an affine transformation. Confocal quadrics in $\mathbb{E}^d$, $d \geq 3$, are characterized by confocal sections with all hyperplanes of symmetry.

(*a*) *Dixon's flexible frameworks*

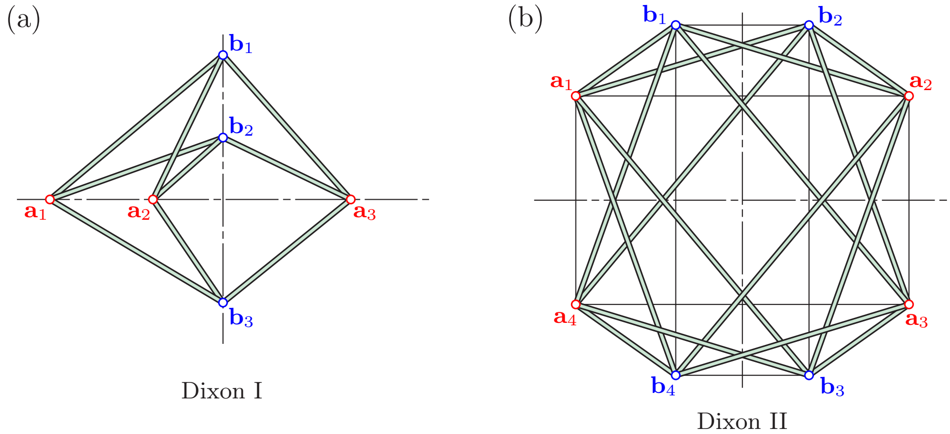

Figure 2. The two types of flexible bipartite frameworks in $\mathbb{E}^2$

A. C. Dixon [3] proved in 1899 that there are exactly two types of continuously flexible bipartite frameworks in the Euclidean plane $\mathbb{E}^2$. A new proof based on algebraic methods can be found in [4].

At type I (Fig. 2a) the two classes of vertices are placed on two orthogonal lines, e.g., the two axes of a cartesian coordinate system. For any given real constant c sufficiently close to 0 the transformation

$$\mathbf{a}_i = (x_i, 0) \mapsto \mathbf{a}'_i = (\sqrt{x_i^2 + c},\, 0), \quad \mathbf{b}_j = (0, y_j) \mapsto \mathbf{b}'_j = (0,\, \sqrt{y_j^2 - c})$$

preserves all distances $\|\mathbf{a}_i - \mathbf{b}_j\| = \sqrt{x_i^2 + y_j^2} = \|\mathbf{a}'_i - \mathbf{b}'_j\|$.

At the second type of flexible bipartite planar frameworks (Fig. 2b) the symmetry is essential: The vertices of two rectangles with two common axes of symmetry constitute the classes of vertices. Again, Ivory's theorem can be used to prove the flexibility (see Fig. 3a):
There is a one-parameter set of conics k passing through $\mathbf{a}_1, \dots, \mathbf{a}_4$. They all have common axes of symmetry. For each k there is a confocal conic k' through $\mathbf{b}_1, \dots, \mathbf{b}_4$. Hence, by Ivory's theorem we can switch to conjugate points thus obtaining a one-parameter set of incongruent realizations of the same framework.

Ivory's theorem is also true on the sphere $\mathbb{S}^2$ (Fig. 3b). Therefore the spherical version of the Dixon-II framework is again flexible. It is called *Bottema's 16-bar framework* [5, 6].

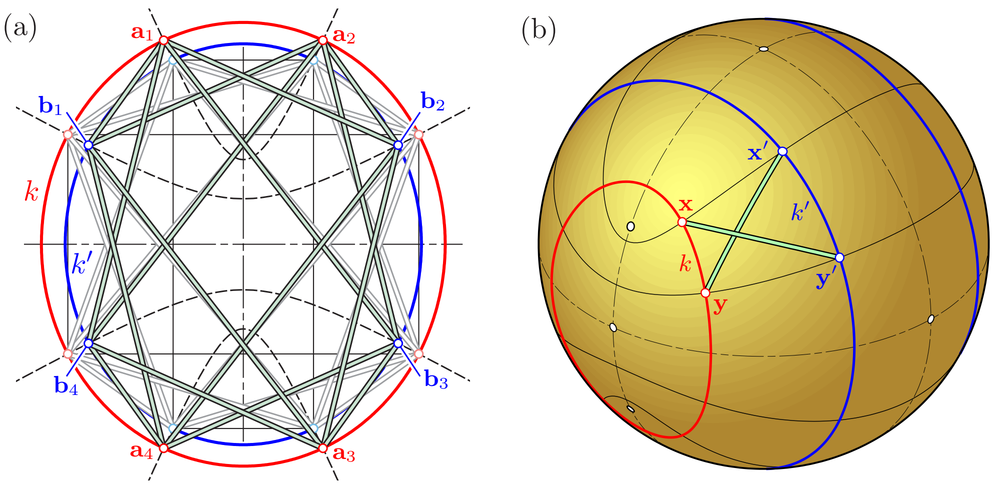

Figure 3. (a) Proving the flexibility of Dixon II by Ivory's theorem; (b): Ivory's theorem on the sphere.

(*b*) *Flexible bipartite frameworks in 3-space*

There is a series of flexible bipartite frameworks in 3-space which can be seen as spatial analoga of the planar Dixon frameworks:

- Let $\mathbf{a}_1, \ldots, \mathbf{a}_{16}$ be corners of two boxes and $\mathbf{b}_1, \ldots, \mathbf{b}_8$ be corners of a third box such that all three boxes are symmetric with respect to a cartesian frame [7].

- The vertices $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are specified on two conics which are located in parallel planes and symmetric with respect to a cartesian frame, while $\mathbf{b}_1, \ldots, \mathbf{b}_8$ are the corners of a box symmetric with respect to the same frame [7].

- $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are located on a conic and $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are arbitrary points in 3-space [8].

- When both classes $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are coplanar and their carrier planes are orthogonal then the degree of freedom of the bipartite framework is at least $k = 3$ [7].

(*c*) *Henrici's flexible hyperboloid and Bricard's octahedra*

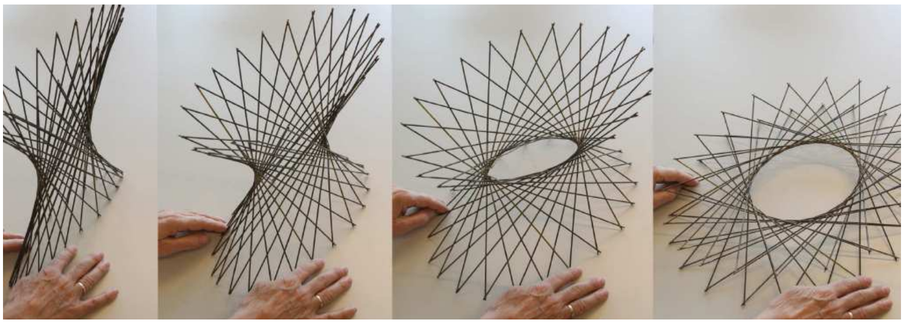

Figure 4. Henrici's flexible hyperboloid (courtesy: G. Glaeser)

At the end of this section we focus on two flexible structures which are related to the bipartite frameworks mentioned before: For any two confocal one-sheet hyperboloids the affinity according to Ivory's theorem (analogue to Fig. 1a) preserves distances along the generators. This is the basis for *Henrici's flexible hyperboloid* [9, 10]:

An arbitrary number of generators of both reguli of an hyperboloid is materialized by rods (see Fig. 4) with a spherical joint at each point of intersection between two rods. In the flat limiting poses the rods are either tangent to the focal hyperbola or tangent to the focal ellipse, the singular surfaces in the range of confocal hyperboloids. Note that this is no more a pure framework but there are hidden constraints since the rods remain aligned during the self-motion.

Due to R. Bricard 1897 [11] there are exactly three types of *flexible octahedra.* Octahedra of type 1 have a line-symmetry (Fig. 5) and those of type 2 a planar symmetry with exactly two vertices located in the plane of symmetry. In both cases the flexibility can be proved using symmetry arguments, only.

Figure 5 shows two particular examples of flexible octahedra of type 1: In both cases the faces $\mathbf{b}_1\mathbf{b}_2\mathbf{a}_2$ and $\mathbf{b}_3\mathbf{b}_4\mathbf{a}_1$ have been omitted in order to avoid

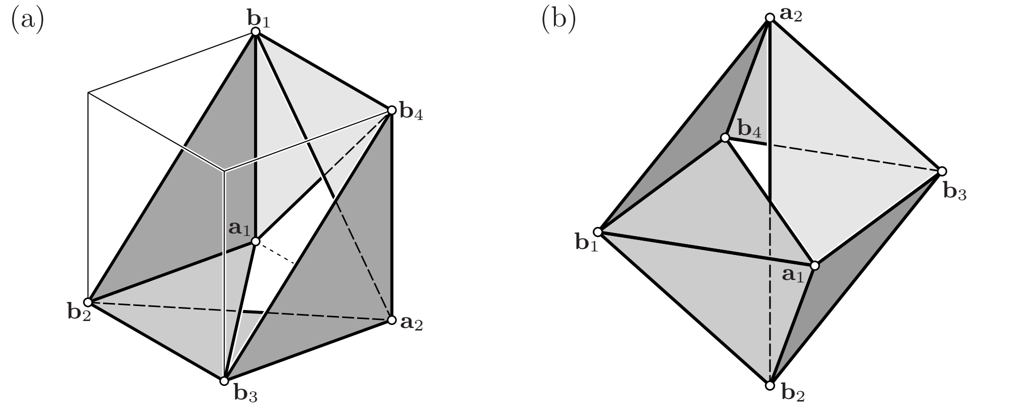

Figure 5. Two particular examples of flexible octahedra where two faces are omitted. Both have an axial symmetry ((a) [12, p. 77]; (b) [13]).

self-intersections. The example (a) is due to W. Blaschke [12]; the displayed pose is related to a cube. The example (b) is related to a regular octahedron and has been found by W. Wunderlich [13]. This example admits two flat poses. In Fig. 6 the unfoldings of these flexible examples are displayed.

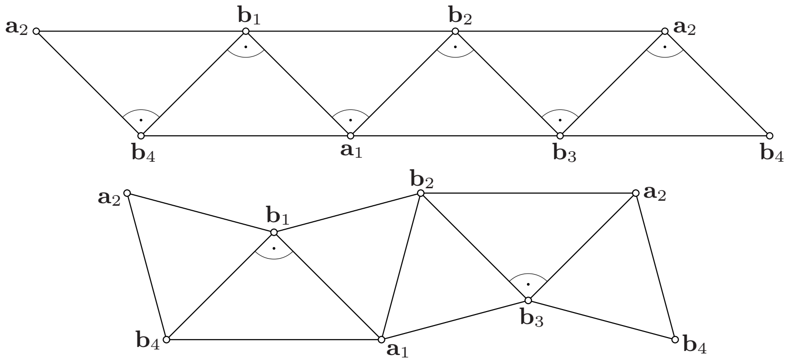

Figure 6. Nets of the flexible octahedra displayed in Fig. 5.

The definition of Bricard's flexible octahedra of type 3, which have no global symmetry, is more complex (see, e.g., [14]). One approach is as follows. The octahedron can be seen as a bipartite framework. The apices $\mathbf{a}_1, \mathbf{a}_2$ of the double-pyramid define one class, the vertices $\mathbf{b}_1, \dots, \mathbf{b}_4$ of the common base quadrangle the other. Therefore again Ivory's theorem can be used to prove that Bricard's examples are the only octahedra which are flexible [15]. G. Nawratil [16] used a different approach when he recently determined all flexible octahedra with one or more vertices at infinity. One type among them is even free of self-intersections.

Figure 7 shows an arbitrary pose $\mathbf{a}_1, \dots, \mathbf{b}_4$ with the quadrangular basis on a one-sheet hyperboloid and as second realization a flat pose $\mathbf{a}'_1, \dots, \mathbf{b}'_4$ where the sides of the quadrangle are tangent to the focal ellipse e' of the hyperboloid. This flat pose cannot be chosen arbitrarily but for a given quadrangle $\mathbf{b}'_1, \dots, \mathbf{b}'_4$

the apices $\mathbf{a}'_1, \mathbf{a}'_2$ must be located on a particular algebraic curve of degree 3, a strophoid [17, Theorem 2]. This curve is the locus of focal points of conics tangent to the sides of $\mathbf{b}'_1, \dots, \mathbf{b}'_4$. The same curve plays a role at Burmester's focal mechanism (note Fig. 11).

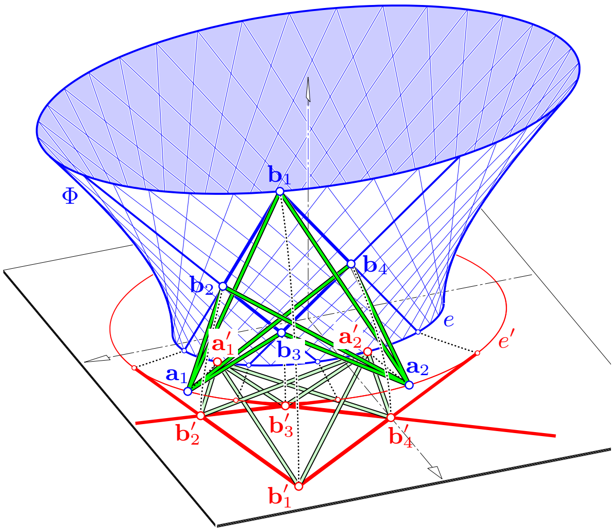

Figure 7. Bricard's flexible octahedron of type 3 and Ivory's theorem

Ivory's theorem is also true in pseudo-Euclidean spaces as well as in spaces of constant curvature [18]. Therefore there are analogues of the flexible bipartite frameworks in other spaces. E.g., the analogues of the three types of flexible octahedra exist also in the hyperbolic 3-space [19].

3. Particular overconstrained flexible mechanisms

Here we present a medley of different flexible structures. Not all are pure frameworks; sometimes there are hidden constraints like collinearities or coplanarities of vertices.

(*a*) *Grünbaum's framework*

The initial pose of *Grünbaum's framework* is highly symmetrical. It admits the full icosahedral group because it consists of the edges of the 10 regular tetrahedra which can be inscribed into a regular pentagon-dodecahedron. With respect to the dodecahedron, the tetrahedra can be devided into two classes, the left ones $\mathrm{L}_1, \dots, \mathrm{L}_5$ and the right ones $\mathrm{R}_1, \dots, \mathrm{R}_5$. We choose the indices such that L_i and R_i are complementary tetrahedra inscribed into the same cube.

Each vertex of the dodecahedron is shared by two tetrahedra. The left L_i contains the left diagonals of the adjacent pentagons, the right R_j contains the right diagonals. We use the ordered pair of indices ij, $i \neq j$, as label of this vertex.

Grünbaum's framework consists of 20 knots and 60 bars. It is at the same time a body-bar framework, but the included ten tetrahedra penetrate each other.

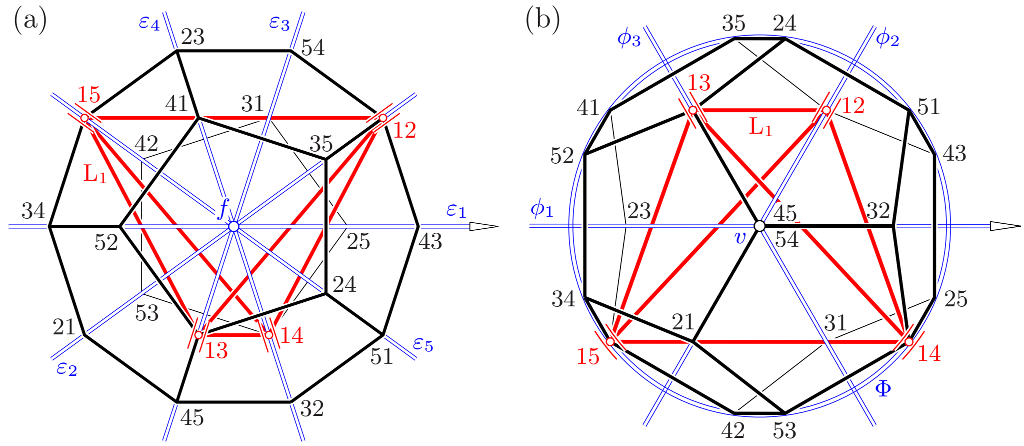

Figure 8. The two types of one-parameter self-motions of Grünbaum's framework

Though eq. (1.1) gives $k = -6$, this framework is flexible. There are two types of one-parameter self-motions which preserve one rotational symmetry:

According to R. Connelly [20] the tetrahedron L_1 can perform a one-parameter motion such that its vertices 12, 13, 14 and 15 remain in the planes $\varepsilon_2, \ldots, \varepsilon_4$ of symmetry through the face axis f, respectively (see Fig. 8a). By iterated rotations about f and reflections in planes through f the movement can be continued to all other tetrahedra of the framework. Fig. 9a shows the traces of the vertices of L_1 under this self-motion which is rational of degree 4 and of type a) according to the classification given in [21].

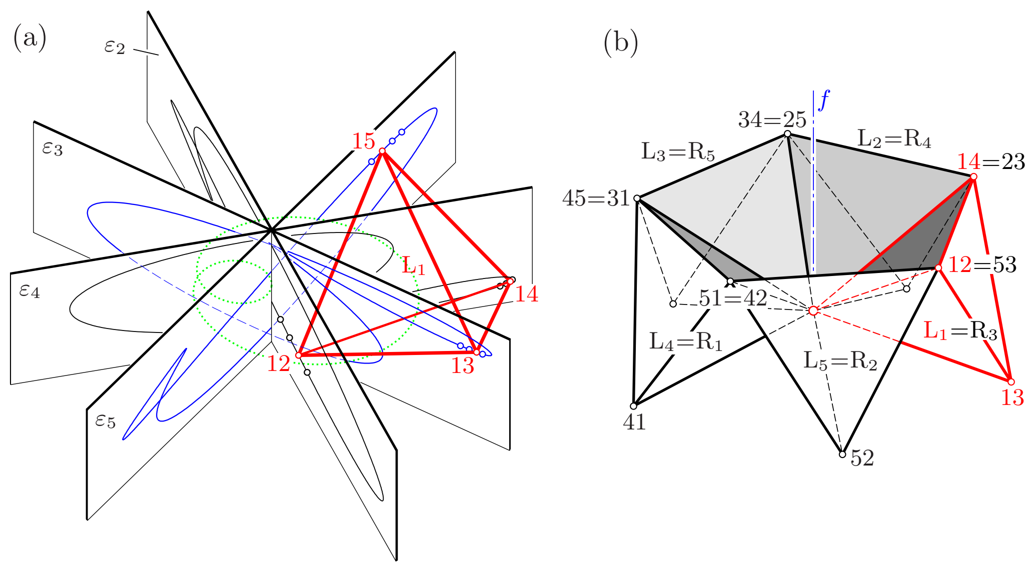

Figure 9. (a) Movement of L_1 under a self-motion of the Grünbaum framework preserving the fivefold symmetry about a face-axis. (b) Pose of bifurcation into a self-motion with d.o.f. = 2.

At a real-world model the movement stops when different vertices come together. However, one can dissolve some joints, and after reassembling the structure the movements continues. The 10 tetrahedra fall apart and form a ring. Then they reach a pose with coinciding pairs of tetrahedra [22]. Here the 5 doubled tetrahedra sit on the faces of a flexible five-sided pyramid without basis (see Fig. 9b). This allows a bifurcation from the one-parameter symmetry-preserving self-motion into a two-parameter motion, which is totally unsymmetric. It is surprising that at this framework the degree of freedom increases by waiving the symmetry. However, it makes sense to see pairweise coinciding tetrahedra as particular case of a local symmetry.

In Fig. 8b another one-parameter self-motion of the Grünbaum framework is displayed [23]: This time the threefold symmetry with respect to a vertex axis v is preserved. The movement of L_1 is such that 12 and 13 remain in the planes ϕ_2, ϕ_3 of symmetry, respectively, while 14 and 15 preserve their distance to the axis v. At the same time the vertices 45 and 54 remain on the axis v which therefore is a fixed axis of symmetry for all poses of L_4, R_4, L_5, and R_5. 120°-rotations about v and reflections in ϕ_1, ϕ_2 and ϕ_3 define the positions of all other tetrahedra.

(*b*) *Chasles' body-bar framework*

This framework consists of two planar bodies Σ, Σ' in $\mathbb{E}^3$ and n connecting bars. The pairwise different anchor points $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \Sigma$ and $\mathbf{a}'_1, \ldots, \mathbf{a}'_n \in \Sigma'$ are projectively related and lie on conics $k \subset \Sigma$ and $k' \subset \Sigma'$, respectively (see Fig. 10). When each achor point represents a spherical joint between bar and body, then a parameter count gives $k = 6 - n$ as degree of freedom.

M. Chasles [24] recognized 1861 with methods of statics, that under these conditions for any n there exists a spatial motion of Σ relative to Σ' which keeps the distances $\|\mathbf{a}_i - \mathbf{a}'_i\|$ fixed for all $i \in \{1, \ldots, n\}$. R. Bricard [25, p. 3, footnote 2] emphasized the kinematic meaning of Chasles' statement — without giving any proof.

We can prove the flexibility by the fact that (in the generic case) the lines connecting corresponding points of two projectively related conics constitute an algebraic ruled surface of degree 4. Such a ruling is always included in a linear line-complex, an argument which is also stressed by K. Wohlhart [26]. The existence of such a linear line-complex is necessary and sufficient for *infinitesimal* flexibility. Since each pose of our framework is infinitesimally flexible, we obtain by integration continuous flexibility, provided there is no stillstand in the initial pose, e.g. caused by two aligned bars.

For $n = 6$ Chasles' mechanism is overconstrained. This is also important for robotics since after replacing the six bars $\mathbf{a}_i\mathbf{a}'_i$ by telescopic legs our framework becomes a particular parallel manipulator, a *planar Stewart Gough platform* (SGP by short). When at a planar SGP all pairs $\mathbf{a}_i \mapsto \mathbf{a}'_i$, $i = 1, \ldots, 6$, are corresponding under a projectivity between two conics k, k', the SGP is singular in each pose, hence *architecturally singular*. A classification of all architecturally singular SGPs has been given by A. Karger [27] in 2003.

The following statement has been presented in [28]: Each planar SGP can be extended by additional legs without restricting its mobility. The method applied in the proof of [28] can also be used here to prove the continuous flexibility of Chasles' framework for any n directly, i.e., not via infinitesimal flexibility.

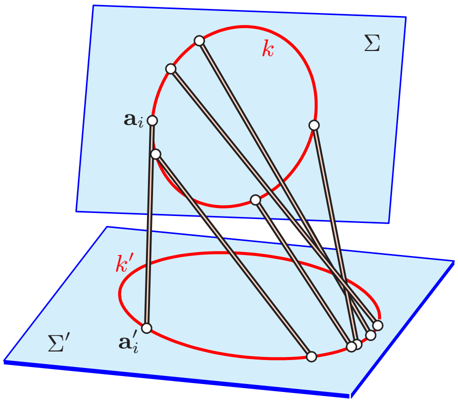

Figure 10. Chasles' body-bar framework; $\mathbf{a}_i \mapsto \mathbf{a}'_i$ is a projectivity between the conics k and k'.

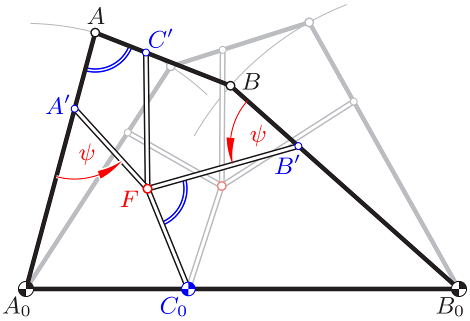

Figure 11. Burmester's focal mechanism; the point triples (A_0, A', A), (B_0, B', B) and (A, C', B) remain collinear during the self-motion.

(*c*) *Burmester's focal mechanism*

For each four-bar A_0B_0BA (see Fig. 11) there are points F such that additional bars connecting F with appropriate intermediate points on the sides do not restrict the flexibility. This has been found 1893 by L. Burmester [29], and he called it a *focal mechanism* since point F must be a focal point of any conic tangent to the sides of the quadrangle A_0B_0BA. The locus of these focal points has already been mentioned before in connection with the flat poses of flexible type-3 octahedra.

A. C. Dixon [3] proved 1899 that the angle $\psi = \sphericalangle A_0A'F$ is congruent to the angle $\sphericalangle BB'F$. For further relations see, e.g., [30, pp. 125–130]. The angles at F are congruent to the interior angles of the quadrangle, e.g., $\sphericalangle C_0FB' = \sphericalangle A_0AB$ (note Fig. 11). Hence the four interior angles in the quadrangle must sum up to 360°, and therefore this mechanism has no spherical analogue! Only the property is preserved on the sphere that under the Dixon condition the composition of the two fourbars A_0C_0FA' and $C_0B_0B'F$ is reducible [31] (compare Fig. 14b).

(*d*) *The Bennett isogram*

Due to G.T. Bennett [32] in 1914 there is a flexible kinematic chain consisting of four cyclically ordered systems $\Sigma_1, \dots, \Sigma_4$ with revolute joints between any two consecutive bodies. When on each revolute axis two points are fixed and each body is replaced by a tetrahedron, we obtain a flexible framework with 8 vertices and 20 bars. Eq. (1.1) gives d.o.f $= 24 - 20 - 6 = -2$.

We give a short proof for its flexibility and start with a skew isogram, i.e., a non-planar quadrangle **abcd** where opposite sides are of equal length,

$$a := \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{c} - \mathbf{d}\|, \quad b := \|\mathbf{b} - \mathbf{c}\| = \|\mathbf{d} - \mathbf{a}\|.$$

We obtain it from a planar parallelogram by bending about one diagonal through the signed angle γ (see Fig. 12a).

Each skew isogram has a *line-symmetry:* A rotation through 180° (half-rotation) about the line m connecting the midpoints of the diagonals exchanges **a** with **c** as well as **b** with **d**. This can be concluded from the congruence of the

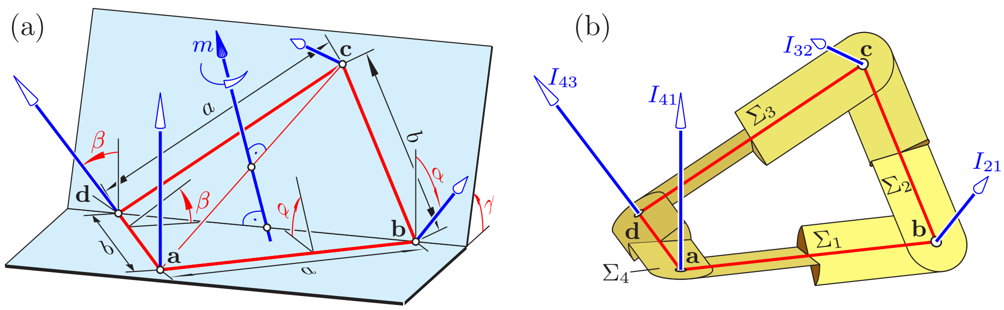

Figure 12. The Bennett isogram as a flexible 4R kinematic chain.

two triangles **dab** and **bcd**. It can also be proved by computation as follows: The sum and the difference of the two equations

$$(\mathbf{a}-\mathbf{b})^2-(\mathbf{c}-\mathbf{d})^2=0 \quad \text{and} \quad (\mathbf{b}-\mathbf{c})^2-(\mathbf{d}-\mathbf{a})^2=0$$

yields after a factorization like $x^2-y^2=(x+y)(x-y)$

$$(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{b}+\mathbf{c}-\mathbf{d})=(\mathbf{b}-\mathbf{d})\cdot(\mathbf{a}-\mathbf{b}+\mathbf{c}-\mathbf{d})=0.$$

This expresses directly that the connection m of the midpoints of the two diagonals is orthogonal to both diagonals.

The convex hull of this skew isogram is a tetrahedron with the basis **dab** and the apex **c**. Let α and β denote the (signed) dihedral angles along the base-edges **ab** and **ad**, respectively (Fig. 12a). When in the congruent triangles **abc** or **cda** the heights on the sides with length a, b are denoted by h_a and h_b, resp., then the height of the apex **c** over the base plane can be expressed in two ways as

$$h_a \sin\alpha = h_b \sin\beta, \quad \text{while} \quad a\,h_a = b\,h_b.$$

Both sides of the second equation give the doubled area of the triangles mentioned before. After elimination of h_a and h_b we get the basic relation

$$a\sin\beta = b\sin\alpha.$$

There is a two-parameter set of mutually incongruent skew isograms sharing the lengths a and b, because the lengths of the diagonals can be chosen independently — within certain limits. We extract a one-parameter set by keeping the dihedral angle α fixed. According to the basic relation β remains constant, too. Since α and β are also the angles between normals of the tetrahedron (see Fig. 12a), our one-parameter set gives flexions of a kinematic revolute-chain with four links (Fig. 12b). Each side of our isogram represents one link Σ_i, when at each endpoint the common perpendicular with the neighboring side serves as the axis of rotation $I_{i\,i+1}$ with respect to the neighbor link Σ_{i+1}.

According to [33] the Bennett isogram is the only flexible 4R kinematic chain. When Σ_1 is kept fixed then point $\mathbf{d}\in\Sigma_4$ remains on a circle with center **a** and axis I_{41} (see Fig. 12b). On the other hand, according to the rotations about I_{21} and I_{32} point $\mathbf{d}\in\Sigma_3$ must be placed on a rotational cyclide [34, 35]. Therefore, the mobility of the Bennett isogram is also related to the existence of different families of circles on rotational cyclides.

Due to J. Krames [36] the relative motion between opposite links is line-symmetric; the axes m of the half-rotations form a regulus of a hyperboloid. The kinematic image of this motion on the Study-quadric is a conic [33, 37]. Therefore this motion serves also as a sort of 'primitive' in motion design [38].

(e) 6R kinematic chains

There is a list of about 30 flexible closed 6R chains including symmetric and non-symmetric ones. They all are overconstrained as eq. 1.1 gives $k=0$.

Recently H.-P. Schröcker [39] could show that the flexibility is related to a factorization of a monic cubic polynomial $P(t)$ over the ring of dual quaternions into linear factors

$$P(t) = (t-h_1)(t-h_2)(t-h_3),$$

where h_1, h_2, h_3 represent rotations. This most interesting result offers a new strategy to find flexible examples, and it reveals clearly that not symmetry but an algebraic property is decisive for continuous flexibility.

4. Flexible Kokotsakis meshes

The following structure is named after Antonios Kokotsakis [40, 41]: A *Kokotsakis mesh* is a polyhedral structure consisting of an n-sided central polygon $\boldsymbol{f}_0$ surrounded by a belt of polygons (Fig. 13). Each side a_i, $i=1,\dots,n$, of $\boldsymbol{f}_0$ is shared by a polygon $\boldsymbol{f}_i$. At each vertex V_i of $\boldsymbol{f}_0$ four faces are meeting.

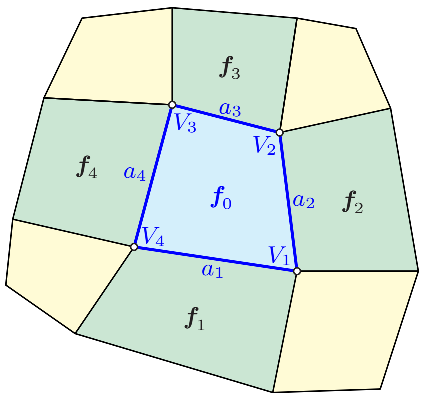

Figure 13. Kokotsakis mesh for $n=4$.

Each face is a rigid body; only the dihedral angles between adjacent faces can vary. An open problem until recently is: Under which conditions a Kokotsakis mesh with $n \geq 4$ is continuously flexible? The answer on this question is of course basic for classifying the flexible compounds of Kokotsakis meshes like *quad meshes* [42] and for *rigid origami*, i.e., for exact paper folding [43]. In the case $n=3$ the posed problem is equivalent to the classification of flexible octahedra (note [16]).

Let us focus on the case $n=4$. In the kinematic sense the polygons $\boldsymbol{f}_0,\dots,\boldsymbol{f}_4$ represent different systems $\Sigma_0,\dots,\Sigma_4$. We keep $\boldsymbol{f}_0 \subset \Sigma_0$ fixed. The sides a_i of $\boldsymbol{f}_0$

are instantaneous axes I_{i0} of the relative motions Σ_i/Σ_0, $i = 1, \ldots 4$. The relative motion Σ_{i+1}/Σ_i between consecutive systems is a spherical four-bar mechanism.

The lengths of the sides $a_1, \ldots, a_4$ of $\boldsymbol{f}_0$ have no influence on the flexibility of the mesh, only their directions. Hence, we can draw the parallels to all axes through a fixed point in space thus obtaining the *spherical image*. A Kokotsakis mesh is flexible if and only if its spherical image is flexible.

The faces meeting at V_i correspond at the spherical image to a spherical four-bar. The interior angles $\alpha_i, \beta_i, \gamma_i, \delta_i$ at V_i are equal to the side lengths of the corresponding four-bar (see Fig. 14). A rotation of $\boldsymbol{f}_1$ about $a_1 = I_{10}$ through the angle φ_1 with respect to $\boldsymbol{f}_0$ corresponds in the spherical image to a rotation of the bar $I_{10}A_1$ about I_{10} through φ_1. The transmission via V_1 to $\boldsymbol{f}_2$ corresponds to the transmission on the sphere via the coupler A_1B_1 to the rotation of the bar $I_{20}B_1$ about I_{20} through φ_2.

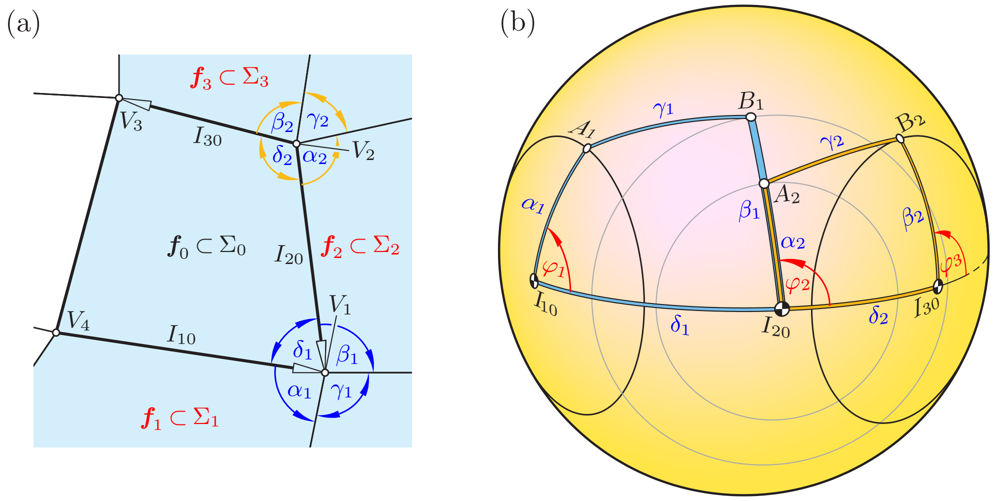

Figure 14. (a) Transmission from $\boldsymbol{f}_1$ to $\boldsymbol{f}_3$ via V_1 and V_2. (b) The composition of two spherical four-bars.

A Kokotsakis-mesh for $n = 4$ is flexible if and only if the transmission from $\boldsymbol{f}_1$ to $\boldsymbol{f}_3$ via V_1 and V_2 shares a component with the transmission via V_4 and V_3. In this case the transmission from the input angle φ_1 to the output angle φ_3 of $\boldsymbol{f}_3$ can be decomposed in two ways by two spherical four-bars. Fig. 15 shows an example. The composition of the four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ is the same as the product of $I_{10}A_1'B_1'I_{20}'$ and $I_{20}'A_2'B_2'I_{30}$. As a control, there is a second pose depicted in lightblue. By the way, the marked angle ψ is the spherical analogue to the Dixon angle ψ marked in Fig. 11.

For $n = 4$ there are 5 types of flexible Kokotsakis-meshes known until recently [44]. In the sequel we use the term *fold* for triples of edges where at each vertex V_i opposite edges are combined. In view of Fig. 13 we can distinguish between two horizontal folds (one includes V_4, V_1, the other through V_2, V_3) and two vertical folds (one through V_4, V_3, the other through V_1, V_2).

I. *Planar-symmetric type:* The Kokotsakis-mesh has a planar symmetry exchanging V_1 with V_4 and V_2 with V_3.

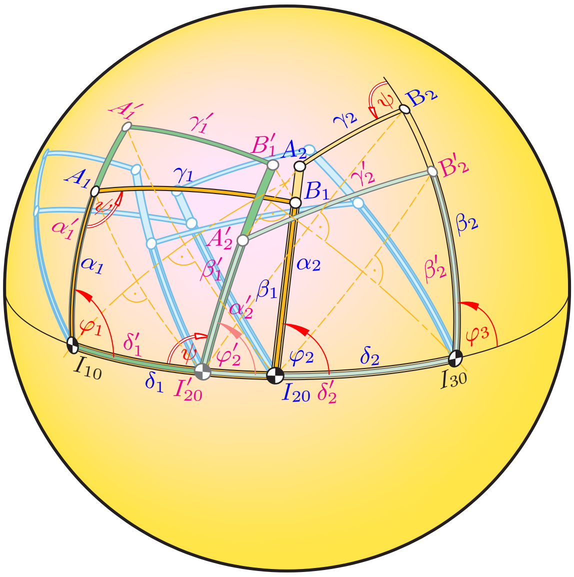

Figure 15. The transmission from φ_1 to φ_3 by the four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ has a common component with that by $I_{10}A_1'B_1'I_{20}'$ and $I_{20}'A_2'B_2'I_{30}$.

II. *Translational type:* There is a translation with $V_1 \mapsto V_4$ and $V_2 \mapsto V_3$ mapping $\boldsymbol{f}_2$ to $\boldsymbol{f}_4$.

III. *Isogonal type* [40]: A Kokotsakis mesh is flexible when at each vertex V_i opposite angles are either equal or complementary, i.e.,

$$\alpha_i = \beta_i, \quad \gamma_i = \delta_i \quad \text{or} \quad \alpha_i = \pi - \beta_i, \quad \gamma_i = \pi - \delta_i.$$

In this case the transmission $\varphi_i \mapsto \varphi_{i+1}$ by the spherical four-bars splits into two linear functions, when expressed in terms of the half angle tangents:

$$\tan\tfrac{\varphi_{i+1}}{2} = f_{i+1\,i}\tan\tfrac{\varphi_i}{2} \quad \text{with} \quad f_{i+1\,i} = \frac{\sin\alpha_i \pm \sin\gamma_i}{\sin(\alpha_i - \gamma_i)}.$$

Additionally, the product of factors must obey $f_{21}f_{32}f_{43}f_{14} = 1$.
This condition is fulfilled as soon as there exists a pose which is non-flat at each each vertex $V_1, \dots, V_4$. A *discrete Voss-surface* is a composition of flexible Kokotsakis meshes of this type [42], e.g., Miura-ori. By the way, also at flexible type-3 octahedra (see Fig. 7) the quadruples of faces at each vertex $\mathbf{a}_1, \dots, \mathbf{b}_4$ form an isogonal pyramid; hence there is a local symmetry though these octahedra are globally unsymmetric.
According to G. Nawratil [45] there is a generalized flexible isogonal type: Even if only at two of the four vertices $V_1, \dots, V_4$ opposite angles are equal or complementary, it can happen that one component of the transmission $\varphi_1 \mapsto \varphi_3$ is decomposable in two ways.

IV. *Orthogonal type* or T-flat [41]: Here the horizontal folds are located in parallel (say: horizontal) planes and the vertical folds in vertical planes.

V. *Line-symmetric type* [44]: A half-rotation maps the pyramid at V_1 onto that of V_4; another one exchanges the pyramids at V_2 and V_3. Additionally

$\delta_1 + \delta_2 = \alpha_1 + \beta_2$ must hold together with the "Dixon angle condition" (note angle ψ at A_1 and B_2 in Fig. 15)

$$\sin\alpha_1\,\sin\gamma_1 : \sin\beta_1\,\sin\delta_1 : (\cos\alpha_1\,\cos\gamma_1 - \cos\beta_1\,\cos\delta_1) =$$
$$\pm\sin\beta_2\,\sin\gamma_2 : \sin\alpha_2\,\sin\delta_2 : (\cos\alpha_2\,\cos\delta_2 - \cos\beta_2\,\cos\gamma_2).$$

Due to Kokotsakis [40, 42] the tessellation of the plane by congruent convex quadrangles, generated by iterated point-reflextions, yields a flexible quad mesh [46]. Here each included Kokotsakis mesh is of type V.

5. Conclusion

We presented a list of overconstrained frameworks and related flexible structures and we explained why they are flexible. Of course, we didn't claim completeness. Personal interest and some common properties were the only motivation why we selected only these examples and no others.

Sometimes the flexibility is a consequence of a symmetry, and the self-motion preserves this symmetry, e.g., at Dixon I, Grünbaum's framework, Bricard's flexible octahedra of types 1 and 2, and the Bennett isogram. However, in the majority of cases particular algebraic properties are responsible for the fact that a structure with particular dimensions is continuously flexible though in the generic case the structure is rigid.

Some of the presented examples date back to the 19th century; others have been detected recently. It turned out that in many cases the flexibility can be concluded from Ivory's theorem. Since this holds for each dimension and in different metrics, the related mechanisms often have analogues in other spaces. But there are also examples without any counterparts in other spaces, e.g., Burmester's focal mechanism.

Generally speaking, there is no 'kings-road' for proving the flexibility of overconstrained structures. Different types require different methods. But this is always a challenge for kinematicians.

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