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The Right-Angle-Theorem in Four Dimensions

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Dedicated to Prof. Walter Wunderlich
at the occasion of his 80th birthday

Abstract

This paper treats perpendicularity in the four dimensional space and its visualization using orthogonal projections onto planes. The elementary approach is based on the formula for the true length of line segments and ends with a characterization of orthogonal views of Cartesian 4D-frames.

Introduction

The well-known right-angle-theorem in three dimensions can be formulated in the following way:

Theorem 1: *Let* $p: E^3 \to \Pi$, $P \mapsto P'$ *be an orthogonal projection of the Euclidean 3-space* E^3 *onto the plane* Π. *Supposing that* a *and* b *are two lines with images* a' *and* b', *each two of the following statements imply the third:*

(i) a *and* b *are perpendicular;*

(ii) a' *and* b' *are perpendicular lines, or either* a' *or* b' *is a point;*

(iii) a *or* b *is parallel to* Π.

Remarks: None of these three implications are valid for oblique projections. On the other hand theorem 1 still holds for orthogonal projections $p: E^n \to E^{n-1}$ in the Euclidean n-space, $n > 3$, since the projection of two especially intersecting lines takes place within a

3-space.

If the dimension of the space of images is smaller than n-1, then (i) and (iii) still imply (ii), but this is not the only way that the right angle is preserved under orthogonal projection.

In the following we deal with parallel projections $p: E^4 \to \Pi$ of the Euclidean 4-space onto planes. Such projections are also the underlying mappings, if parallel projections $E^4 \to E^3$ are composed with parallel projections $E^3 \to \Pi$, to say if e.g. the 3-dimensional image of a 4-space-object is displayed on a sheet of paper or on the screen (see e.g. Seiner and Slaby, 1988 or Zuji Wan, Qidi Lin and Duane, 1989). The center of projection p is a line at infinity; there are completely parallel planes of sight.*)

Mating orthogonal projections of E^4

Let (x_1, x_2, x_3, x_4) and (x,y) be Cartesian coordinates in E^4 and E^2 respectively. Then the mappings

$$\alpha_1: E^4 \to E^2,\ P = (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto P' = (\xi_1, \xi_2),$$

$$\alpha_2: E^4 \to E^2,\ P = (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto P'' = (-\xi_3, -\xi_4)$$

give a bijection $E^4 \to (E^2 \times E^2)$, $P \mapsto (P', P'')$.

α_1 can be decomposed into the mapping

$$p_1: E^4 \to \Pi_1,\ (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_1, \xi_2, 0, 0)$$

and a congruence

$$\Pi_1 \to E^2,\ (\xi_1, \xi_2, 0, 0) \mapsto (\xi_1, \xi_2).$$

p_1 turns out to be an orthogonal projection; its

*) For two different points in the same plane of sight it is impossible to claim that one lies before the other. Hence it doesn't make sense to distinguish between visible and invisible edges in two dimensional views of the 4-space (cf. Laghi, 1988).

planes of sight under p_1 are parallel to the plane $\Pi_2: x_1 = x_2 = 0$, since $(\xi_1, \xi_2, 0, 0)$ is the point of intersection of Π_1 with the plane of sight $x_1 = \xi_1$, $x_2 = \xi_2$ passing through P. Π_2 is completely perpendicular to Π_1.

Analogously α_2 is the product of the orthogonal projection

$$p_2: E^4 \to \Pi_2, \ (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (0, 0, \xi_3, \xi_4)$$

and the congruence $\Pi_2 \to E^2$, $(0, 0, \xi_3, \xi_4) \mapsto (-\xi_3, -\xi_4)$.

The mappings α_1, α_2 with mutually perpendicular planes of sight are called ***mating orthogonal projections*** of the 4-space. Such pairs of mappings have e.g. been used in Shoute, 1902, Eckhart, 1929, Hohenberg and Tschupik, 1971. The congruences $\Pi_i \to E^2$ can be arbitrarily selected. We made the particular choice above just for convenience.

The true distance $\overline{PQ}$ of $P = (\xi_1, \dots)$ and $Q = (\eta_1, \dots)$ matches the equation (see fig. 1)

$$\text{(1.1)} \qquad \overline{PQ}^2 = \overline{P'Q'}^2 + \overline{P''Q''}^2,$$

since

$$\overline{PQ}^2 = [(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2] + [(\xi_3 - \eta_3)^2 + (\xi_4 - \eta_4)^2].$$

Now we state a right-angle-theorem in the 4-space:

Theorem 2: *Let* α_1, α_2 *be mating orthogonal projections* $P \mapsto (P', P'')$. *Supposing that* a, b *are two lines with images* a', b' *and* a'', b'' *respectively, each two of the following statements imply the third:*

(i) a *and* b *are perpendicular;*

(ii) *either* a' *and* b' *are perpendicular lines, or* a' *or* b' *is a point;*

(iii) *either* a'' *and* b'' *are perpendicular lines, or* a'' *or* b'' *is a point.* *)

Proof: We can assume that a and b are intersecting. Let ABC be a triangle with $A, C \in a$ and $B, C \in b$. Then we have the following equivalences:

(i) $\iff$ $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$;

(ii) $\iff$ $\overline{A'B'}^2 = \overline{A'C'}^2 + \overline{B'C'}^2$;

(iii) $\iff$ $\overline{A''B''}^2 = \overline{A''C''}^2 + \overline{B''C''}^2$.

Now due to (1.1) the sum or the difference of each two equations on the right side implies the third.

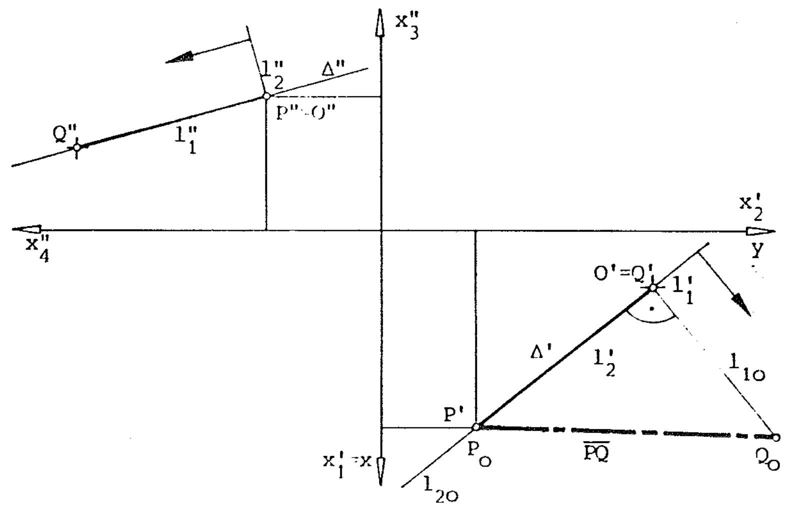

Figure 1

In contrast with the 3-space the 4D-version of the right-angle-theorem also holds for particular oblique projections. For given angles φ_1, φ_2, $\pi > \varphi_i > 0$, let us replace the image planes Π_1 of p_1 and Π_2 of p_2 by planes Ψ_1, Ψ_2,

*) If a' is a point, a'' necessarily is a line.

$$\Psi_i: x_1 \cos\varphi_i - x_3 \sin\varphi_i = x_2 \cos\varphi_i - x_4 \sin\varphi_i = 0,$$

$i = 1,2$, while the planes of sight remain unchanged. The unit vectors

$e_1 = (\sin\varphi_1, 0, \cos\varphi_1, 0)$ and $e_2 = (0, \sin\varphi_1, 0, \cos\varphi_1)$ in Ψ_1 and

$e_3 = (\cos\varphi_2, 0, \sin\varphi_2, 0)$ and $e_4 = (0, \cos\varphi_2, 0, \sin\varphi_2)$ in Ψ_2 determinate Cartesian coordinate systems with origin $(0,0,0,0)$ in both planes. We calculate

$$p_1: (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_1, \xi_2, \xi_1 \cot\varphi_1, \xi_2 \cot\varphi_1) = (\xi_1 e_1 + \xi_2 e_2)/\sin\varphi_1;$$

$$p_2: (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_3 \cot\varphi_2, \xi_4 \cot\varphi_2, \xi_3, \xi_4) = (\xi_3 e_3 + \xi_4 e_4)/\sin\varphi_2.$$

Hence the two images of $P = (\xi_1, \xi_2, \xi_3, \xi_4)$ with plane coordinates $(\xi_1/\sin\varphi_1, \xi_2/\sin\varphi_1)$ in Ψ_1 and $(\xi_3/\sin\varphi_2, \xi_4/\sin\varphi_2)$ in Ψ_2 are related to the initial images P', P" by dilatations with factor $1/\sin\varphi_i \geq 1$. For $i=1,2$ the plane Ψ_i makes equal angles φ_i with the corresponding planes of sight. Therefore these particular parallel projections should be called *isocline* (see Manning, 1956, p.123), including orthogonal projections under $\varphi_i = \pi/2$.

Remark: Theorem 2 holds also for two "complementary" orthogonal projections

$$p_1: E^n \to E^m, P \mapsto P' \text{ and } p_2: E^n \to E^{n-m}, P \mapsto P''$$

for $E^m \perp E^{n-m}$, $1 < m < n-1$, $n>4$. The proof is based on the fact that for each two vectors **a**, **b** the dot-product matches the equation **a.b** = **a'.b'** + **a".b"**.

Perpendicularity of planes in E^4

The image of a plane Ψ under the mapping α_1 is a point, if and only if Ψ is completely perpendicular to Π_1. Planes Ψ that are half perpendicular to Π_1

appear in an edge view. For all other planes the mapping $\Psi \to E^2$, $P \mapsto P'$ is a regular affinity.

A plane Δ being half perpendicular to both Π_1 and Π_2 is ***called double-projecting.*** Δ' and Δ'' are lines and we have

$$P \in \Delta \iff P' \in \Delta' \text{ and } P'' \in \Delta''.$$

The plane Δ contains lines l_1 perpendicular to Π_1 and lines l_2 perpendicular to Π_2. Due to theorem 2 the lines l_1 and l_2 are mutually perpendicular. This gives raise to a method for finding the true shape of figures lying in Δ: We define Cartesian coordinates in Δ with axes parallel to l_1, l_2 and with an arbitrary origin O. According to (1.1) each view shows one coordinate in true size. We construct the true shape by assembling the two coordinates for each point Q of Δ (see fig. 1).

Let Δ_1 be a double-projecting plane and let Δ_2 be completely perpendicular to Δ_1. Then Δ_2 must be double-projecting too and in both views the images Δ_1', Δ_2' and Δ_1'', Δ_2'' must be perpendicular. This can be deduced from theorem 2: We suppose that $l_1, l_2 \subset \Delta_1$ are perpendicular to Π_1, Π_2 respectively. Then the second view of each line perpendicular to l_1 must be either a point or perpendicular to l_1''. The same holds for the first view of each line perpendicular to l_2.

The double-projecting planes Δ_1 and Δ_2 can be used as image planes of new mating orthogonal projections. We find these auxiliary views after constructing the true shape of Δ_1 and Δ_2 (cf. Lindgren and Slaby, 1968).

Now let Ψ be a plane that is neither double-projecting nor completely perpendicular to Π_1 or Π_2: The (regular or singular) affinity $\Psi \to E^2$ with $P \mapsto P'$ maps an unit circle $c \subset \Psi$ onto an ellipse (or a line segment) c' (see fig. 2). Its axes m', n' of symmetry are images of perpendicular diameters m, n of c. Due to theorem 2 the lines m" and n" must be perpendicular too. If the semiaxes of c' are denoted by $\cos\mu$ and $\cos\nu$, then according to (1.1) the semiaxes of c" read $\sin\mu$ and $\sin\nu$. Here μ and ν are the two angles made by Ψ and Π_1.

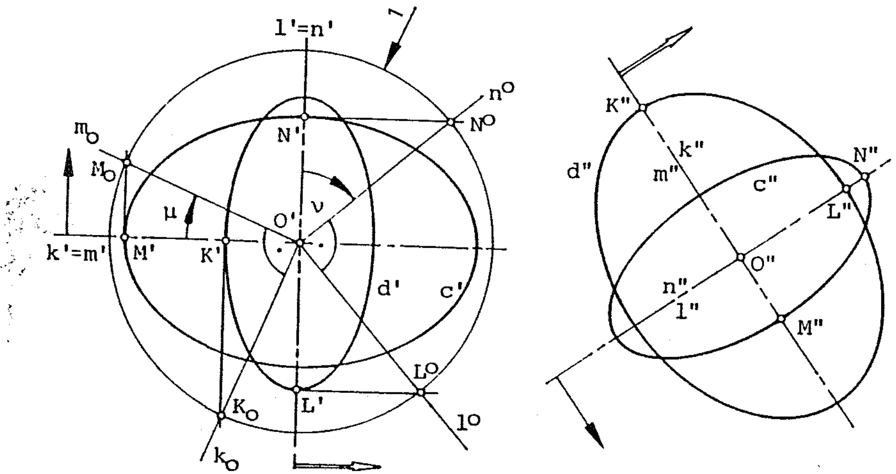

Figure 2

Let Φ be the plane completely perpendicular to Ψ that passes through the center O of c. We span this plane by lines k and l through O, that are perpendicular to both m and n. We select line k perpendicular to m within the double projecting plane through m; then it is also perpendicular to n

because of theorem 2. Analogously l is defined by l'=n' and l"=n". Due to theorem 2 k and l are mutually perpendicular. The unit circle $d \subset \Phi$ with center O is mapped on ellipses (or line segments) d' and d", symmetric with respect to the images of k and l. The true shapes of the double projecting planes through m and n reveal that d' must be congruent to c" and vice versa. All four ellipses have the same excentricity.

Theorem 3: *Five points* U', A', B', C', D' *in a plane* Π *are the images of origin* U *and unit points* A,...,D *of a 4D-Cartesian frame under orthogonal projection if and only if the two ellipses*) with conjugate diameters* U'A', U'B' *and* U'C', U'D' *respectively have focal points that are corresponding under a 90°-rotation about* U'.
The edge length $e = \overline{UA}$ *of the frame is given by*

$$\overline{U'A'}^2 + \overline{U'B'}^2 + \overline{U'C'}^2 + \overline{U'D'}^2 = 2e^2.$$

Proof: We can appoint the two ellipses as the first images of two congruent and concentric circles c, d in completely perpendicular planes. The second views c", d" must be congruent to d', c' respectively. The given conjugate diameters U'A', U'B' of c' and U'C' and U'D' of d' are therefore images of mutually perpendicular diameters of c and d and they define a Cartesian frame in E^4. It is wellknown

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*) If U', A', B' are aligned, then the ellipse c is in edge view. The length 2l of the line-segment c' is given by $l^2 = \overline{U'A'}^2 + \overline{U'B'}^2$; the focal points of c' coincide with its endpoints.

that the semiaxes $e.\cos\mu$, $e.\cos\nu$ of c' and $e.\sin\mu$, $e.\sin\nu$ of d' match the equations

$$\overline{U'A'}^2 + \overline{U'B'}^2 = e^2(\cos^2\mu + \cos^2\nu)$$
$$\overline{U'C'}^2 + \overline{U'D'}^2 = e^2(\sin^2\mu + \sin^2\nu)$$

and this results in the e defining equation given in theorem 3 (cf. Stachel, 1987).

Let $a,\ldots,d \in C$ be the complex coordinates $x+iy$ of A', ..., D' with respect to a coordinate system in Π with origin U'. Then due to Stiefel, 1938, the complex coordinates $\pm\gamma$, $\pm\delta$ of the focal points of the ellipses c' and d' are given by

$$\gamma^2 = a^2 + b^2, \ \delta^2 = c^2 + d^2,$$

and from $\delta = \pm i\gamma$ we derive the 4D-version of the Gaussian formula

$$a^2 + b^2 + c^2 + d^2 = 0.$$

This shows that the relation between the four line segments U'A', U'B', U'C', U'D' is symmetric. m-dimensional orthogonal views of n-dimensional Cartesian frames have already been characterized by Naumann, 1957 (see Stachel, 1987).

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