FLEXIBLE OCTAHEDRA IN THE HYPERBOLIC SPACE

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Abstract This paper treats flexible polyhedra in the hyperbolic 3-space $\mathbb{H}^3$. It is proved that the geometric characterization of octahedra being infinitesimally flexible of orders 1 or 2 is quite the same as in the Euclidean case. Also Euclidean results concerning continuously flexible octahedra remain valid in hyperbolic geometry: There are at least three types of continuously flexible octahedra in $\mathbb{H}^3$; the line-symmetric Type 1, Type 2 with planar symmetry, and the non-symmetric Type 3 with two flat positions. However, Type 3 can be subdivided into three subclasses according to the type of circles in hyperbolic geometry. The flexibility of Type 3 octahedra can again be argued with the aid of Ivory's Theorem.

Keywords: Flexible polyhedra, Bricard's octahedra, infinitesimal flexibility, hyperbolic geometry

Introduction

R. Bricard's *continuously flexible* octahedra ([1], compare also [9, 5]) play an essential role in the theory of flexible polyhedra. The first two types of flexible octahedra in the Euclidean 3-space $\mathbb{E}^3$ admit self-symmetries: All pairs of opposite vertices of *Type 1* are symmetric with respect to a line; at *Type 2* two pairs of vertices are symmetric with respect to a plane which passes through the remaining two vertices. Octahedra of *Type 3* are unsymmetric and admit two flat positions which in a certain way are related to three concentric circles (see e.g. [7]). Bricard proved in [1] that these three types are the only octahedra in $\mathbb{E}^3$ which are continuously flexible — apart from two trivial cases which either have one equator aligned or two opposite vertices coinciding.

Beside the continuously flexible exemplars also the *infinitesimally flexible* octahedra deserve interest. They can be classified with respect to the *order* $n \geq 1$ of flexibility. In [6] geometric characterizations were given for octahedra which are flexible either of first or of second order (compare also [2, 9]).

The aim of this paper is to demonstrate that these characterizations remain valid in the hyperbolic space $\mathbb{H}^3$ (Theorem 1). Furthermore, it will be proved (Theorem 3) that the hyperbolic counterparts of Bricard's octahedra are again continuously flexible.

The flexibility of Types 1 and 2 in $\mathbb{H}^3$ can be proved like in $\mathbb{E}^3$: Let a skew isogram $B_1C_1B_2C_2$ be given, i.e., a quadrangle with the property that *opposite* sides have equal length. Then each pair (B_1, B_2) and (C_1, C_2) is symmetric with respect to an axis a.[1] Any arbitrary point A_1 can serve as a vertex for a pyramid with basis $B_1C_1B_2C_2$. This pyramid consisting of four triangles is flexible in $\mathbb{H}^3$. And this flexibility is not restricted when we add its mirror under reflection in the axis a. Of course, at each Bricard's octahedron we have to neglect self-intersections.

When a quadrangle $B_1C_1B_2C_2$ is given where two pairs of *neighboring* sides are of equal length, e.g., $d_h(B_1, C_i) = d_h(B_2, C_i)$, $i = 1, 2$, then the vertices B_1, B_2 are symmetric with respect to a plane through C_1 and C_2. In a similar way as before two symmetric pyramids with the common basis $B_1C_1B_2C_2$ constitute a flexible octahedron which is of Type 2.

The description of octahedra of Type 3 (with flat positions) is more complicated and will be given in Section 2 below.

It is conjectured that these three types are the only nontrivial examples of flexible octahedra in $\mathbb{H}^3$. However, a complete proof is still open. For the Euclidean case Bricard's main result in [1] has been reproved in [5] with methods from projective geometry. The proof was based on a configuration theorem concerning bipartite frameworks (see [8]). The hyperbolic counterpart of this theorem has not yet been proved.

Most of the following statements are based on the projective model of $\mathbb{H}^3$ with the absolute quadric Ω. We use a coordinate system in the real projective 3-space $\mathbb{P}^3$ such that for any two points $X = \mathbf{x}\mathbb{R}$, $Y = \mathbf{y}\mathbb{R}$ conjugate position with respect to Ω is equivalent to

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 = 0. \tag{1}$$

Then for points[2] $\mathbf{x}, \mathbf{y} \in \mathbb{H}^3$ the coordinates can be normalized to $\langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = 1$, and their hyperbolic distance $d_h(\mathbf{x}, \mathbf{y})$ obeys

$$\cosh d_h(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle, \quad \text{provided} \quad \langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = 1. \tag{2}$$

[1] The triangles $B_1B_2C_1$ and $B_2B_1C_2$ are congruent. Therefore there is a product of reflections in two perpendicular planes with $B_1 \mapsto B_2$, $C_1 \mapsto C_2$, and vice versa.

[2] In the sequel we often identify the point X with any of its coordinate vectors $\mathbf{x}$ when we briefly speak about 'point' $\mathbf{x}$.

This is the so-called Weierstraß model of $\mathbb{H}^3$. It is located on the unit sphere of the four-dimensional Minkowski space $\mathbb{M}^4$.

1. Infinitesimally flexible octahedra in $\mathbb{H}^3$

Theorem 1 *Let $\mathcal{O}$ be an octahedron in $\mathbb{H}^3$ with the non-coplanar 'equator' $B_1C_1B_2C_2$ and the 'poles' $A_1 \neq A_2$.*

1. *$\mathcal{O}$ is infinitesimally flexible of first order if and only if there is a second-order surface Φ_1 passing through the vertices A_1, A_2 and through the sides of the equator $B_1C_1B_2C_2$.*
2. *A first-order infinitesimally flexible octahedron $\mathcal{O}$ with surface Φ_1 according to 1. is infinitesimally flexible of order two if and only if there are second-order surfaces Φ_{2a} through the poles A_1, A_2 and Φ_{2b} through the sides of the equator $B_1C_1B_2C_2$ such that the pencil spanned by Φ_{2a} and Φ_{2b} includes the surface Ψ_2 which is polar to the absolut quadric Ω with respect to Φ_1.*

Proof. In analogy to the Euclidean definition (cf. [3, 4]) a framework $\mathcal{F}$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and edges $e_{jk} = \mathbf{v}_j\,\mathbf{v}_k$, $(i,j) \in E$, is called *infinitesimally flexible of order* n (in the classical sense) in $\mathbb{H}^3$ if and only if for each $i \in \{1, \ldots, n\}$ there is a polynomial function

$$\mathbf{v}_i' := \mathbf{v}_i + \mathbf{v}_{i,1}t + \ldots + \mathbf{v}_{i,n}t^n, \quad n \geq 1, \tag{3}$$

such that

(a) the replacement of $\mathbf{v}_i$ by $\mathbf{v}_i' \in \mathbb{R}[t]^4$ in the formulas for the edge lengths gives stationary values of multiplicity $\geq n$ at $t = 0$, i.e., due to (2)

$$\begin{array}{l} \langle\, \mathbf{v}_j', \mathbf{v}_k' \rangle - \langle\, \mathbf{v}_j, \mathbf{v}_k \rangle = o(t^n) \quad \forall (j,k) \in E, \\ \text{while } \ \langle\, \mathbf{v}_i', \mathbf{v}_i' \rangle - 1 = o(t^n) \quad \forall i \in \{1, \ldots, n\}. \end{array} \tag{4}$$

(b) In order to exclude *trivial* flexes, the vectors $\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{n,1}$ do not originate from any motion of $\mathcal{F}$ as a rigid body.

The n-tupel $(\mathbf{v}_1', \ldots, \mathbf{v}_n')$ of polynomial vector functions is called a non-trivial *n-th-order flex* of $\mathcal{F}$.

Conditions for n-th order flexibility of $\mathcal{O}$. The 12 edges of the octahedron $\mathcal{O}$ define a framework in $\mathbb{H}^3$ with 6 vertices $\mathbf{a}_1, \ldots, \mathbf{c}_2$. We change the notation of the equator slightly by setting

$$\mathbf{v}_1 := \mathbf{b}_1,\ \mathbf{v}_2 := \mathbf{c}_1,\ \mathbf{v}_3 := \mathbf{b}_2,\ \mathbf{v}_4 := \mathbf{c}_2\,.$$

Now in analogy to [6] we subdivide the edge set of $\mathcal{O}$ into the equator $\{\mathbf{v}_1\mathbf{v}_2, \ldots, \mathbf{v}_4\mathbf{v}_1\}$ and the 8 sides $\mathbf{v}_j\mathbf{a}_k$, $j \in \{1, \ldots, 4\}$, $k \in \{1, 2\}$. The latter form a bipartite sub-framework $\mathcal{O}'$ of $\mathcal{O}$.

Let an n-th-order flex of $\mathcal{O}$ be given by

$$\mathbf{v}'_j = \mathbf{v}_j + \mathbf{v}_{j,1}t + \ldots + \mathbf{v}_{j,n}t^n, \ \mathbf{a}'_k = \mathbf{a}_k + \mathbf{a}_{k,1}t + \ldots + \mathbf{a}_{k,n}t^n \tag{5}$$

such that

$$\begin{aligned} &\langle\, \mathbf{v}'_j,\, \mathbf{a}'_k \rangle - \langle\, \mathbf{v}_j,\, \mathbf{a}_k \rangle = o(t^n), \quad \langle\, \mathbf{v}'_j,\, \mathbf{v}'_{j+1} \rangle - \langle\, \mathbf{v}_j,\, \mathbf{v}_{j+1} \rangle = o(t^n), \\ &\langle\, \mathbf{v}'_j,\, \mathbf{v}'_j \rangle - \langle\, \mathbf{v}_j,\, \mathbf{v}_j \rangle = o(t^n), \quad \langle\, \mathbf{a}'_k,\, \mathbf{a}'_k \rangle - \langle\, \mathbf{a}_k,\, \mathbf{a}_k \rangle = o(t^n) \end{aligned} \tag{6}$$

for all $j \in \{1, \ldots, 4\}$ and $k \in \{1, 2\}$. From now on we assume a *non-coplanar* equator $\mathbf{v}_1 \ldots \mathbf{v}_4$. Then at each $t \in \mathbb{R}$ there is a linear mapping

$$l(t)\colon\ \mathbb{R}^4 \to \mathbb{R}^4, \quad \mathbf{v}_j \mapsto \mathbf{v}'_j(t) \ \text{ for } \ j = 1, \ldots, 4.$$

For each $k \in \{1, 2\}$ the equations

$$\langle\, l(\mathbf{v}_j),\, \mathbf{a}'_k \rangle - \langle\, \mathbf{v}_j,\, \mathbf{a}_k \rangle = o(t^n)$$

define a system of four linearly independent equations for the unknown vector $\mathbf{a}_k$. Let the mapping $l^\star$ be adjoint to l, i.e., obeying $\langle\, l(\mathbf{x}),\, \mathbf{y} \rangle = \langle\, \mathbf{x},\, l^\star(\mathbf{y}) \rangle$ for each $t \in \mathbb{R}$. Then the solution of the linear system can be expressed as

$$\mathbf{a}_k = l^\star(\mathbf{a}'_k) + \mathbf{o}(t^n) \ \text{ for } \ k = 1, 2.$$

Thus the first equation of (6) holds true for all edges of $\mathcal{O}'$. For t sufficiently near to 0 the linear mapping l is bijective as well as $l^\star$. Here we introduce two bilinear forms over $\mathbb{R}[t]$,

$$\begin{aligned} f(t; \mathbf{x}, \mathbf{y}) &:= \langle\, l(\mathbf{x}),\, l(\mathbf{y}) \rangle - \langle\, \mathbf{x},\, \mathbf{y} \rangle, \\ g(t; \mathbf{x}, \mathbf{y}) &:= \langle\, \mathbf{x},\, \mathbf{y} \rangle - \langle\, l^{\star -1}(\mathbf{x}),\, l^{\star -1}(\mathbf{y}) \rangle. \end{aligned} \tag{7}$$

Thus we can summarize the remaining conditions in (6) for an n-th order infinitesimal flex as

$$f(\mathbf{v}_j, \mathbf{v}_j) = o(t^n), \quad f(\mathbf{v}_j, \mathbf{v}_{j+1}) = o(t^n), \quad g(\mathbf{a}_k, \mathbf{a}_k) = o(t^n). \tag{8}$$

These are homogeneous in $\mathbf{v}_j$ and $\mathbf{a}_k$.

Now we turn over to matrix notation. We write the coordinate vectors as columns and set up

$$l(t)\colon\ \mathbf{v}_j \ \mapsto \ \mathbf{v}'_j = A(t)\mathbf{v}_i \ \text{ with } \ A(0) = I_4\,, \tag{9}$$

where I_4 denotes the $4{\times}4$ unit matrix. The entries of A are polynomials of degree $\leq n$. Let H denote the diagonal matrix $\mathrm{diag}(1, -1, -1, -1)$

with $H^{-1} = H^T = H$. Then the fundamental bilinear form (1) can be expressed as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T H \mathbf{y},$$

and the mapping adjoint to l reads

$$l^\star(t):\ \mathbf{a}'_k \mapsto \mathbf{a}_k = H A^T H \mathbf{a}'_k\,. \tag{10}$$

Therefore the coordinatizations of the bilinear forms defined in (7) are

$$\begin{aligned} f(t;\mathbf{x},\mathbf{y}) &:= \mathbf{x}^T(A^T H A - H)\mathbf{y},\\ g(t;\mathbf{x},\mathbf{y}) &:= \mathbf{x}^T(H - H A^{-1} H A^{T^{-1}} H)\mathbf{y}. \end{aligned} \tag{11}$$

Lemma 1 *The octahedron $\mathcal{O}$ with non-coplanar equator $\mathbf{v}_1 \ldots \mathbf{v}_4$ is infinitesimally flexible of order n if and only if in a neighborhood of $t = 0$ there is a regular matrix $A(t)$ with entries of class C^n and $A(0) = I_4$ such that the vertices $\mathbf{v}_1, \ldots, \mathbf{a}_2$ obey the equations* (8) *with bilinear forms f, g according to* (11) *— provided the flex corresponding to $\mathbf{v}'_j(t) = A\mathbf{v}_j$ and $\mathbf{a}'_k(t) = H A^{T^{-1}} H \mathbf{a}_k$ is not trivial.*

In order to obtain geometric characterizations for n-order flexible octahedra, $n \in \{1, 2, \ldots\}$, we compare the coefficients of t^n in the equations (8). For this purpose we use the Taylor expansions (compare [7])

$$\begin{aligned} A(t) &= I_4 + A_1 t + A_2 t^2 + \ldots\,,\\ A^{-1}(t) &= I_4 - A_1 t + (-A_2 + A_1^2)t^2 + \ldots\,. \end{aligned} \tag{12}$$

This implies for the bilinear forms (11)

$$\begin{aligned} &A^T H A - H = (A_1^T H + H A_1)t + (A_2^T H + A_1^T H A_1 + H A_2)t^2 + \ldots\,,\\ &H - H A^{-1} H A^{T^{-1}} H \ = \ (A_1^T H + H A_1)t \ +\\ &+ (A_2^T H - A_1^{T^2} H - H A_1 H A_1^T H - H A_1^2 + H A_2)t^2 + \ldots\,. \end{aligned}$$

We set

$$\begin{aligned} f(t;\mathbf{x},\mathbf{y}) &= f_1(\mathbf{x},\mathbf{y})t + f_2(\mathbf{x},\mathbf{y})t^2 + \ldots\,,\\ g(t;\mathbf{x},\mathbf{y}) &= g_1(\mathbf{x},\mathbf{y})t + g_2(\mathbf{x},\mathbf{y})t^2 + \ldots \end{aligned} \tag{13}$$

and obtain

$$f_1(\mathbf{x},\mathbf{y}) = g_1(\mathbf{x},\mathbf{y}) = \mathbf{x}^T(H A_1 + A_1^T H)\mathbf{y}, \tag{14}$$

$$\begin{aligned} f_2(\mathbf{x},\mathbf{y}) &= \mathbf{x}^T(A_2^T H + A_1^T H A_1 + H A_2)\mathbf{y},\\ g_2(\mathbf{x},\mathbf{y}) &= \mathbf{x}^T(A_2^T H - A_1^{T^2} H - H A_1 H A_1^T H - H A_1^2 + H A_2)\mathbf{y}. \end{aligned} \tag{15}$$

First-order flexibility. Suppose, the bilinear form f_1 in (14) is not zero. Then $f_1(\mathbf{x},\mathbf{x}) = 0$ is the equation of a second-order surface Φ_1. Due to

$$f_1(\mathbf{v}_i,\mathbf{v}_i) = f_1(\mathbf{v}_i,\mathbf{v}_{i+1}) = f_1(\mathbf{a}_k,\mathbf{a}_k) = 0$$

this surface Φ_1 passes through the equator and through the poles.

According to Lemma 1 and (12) the 'velocity vectors' under this flex in the Minkowski space $\mathbb{M}^4$ are

$$\mathbf{v}_{i,1} = A_1\mathbf{v}_i, \quad \mathbf{a}_{k,1} = -HA_1^T H\mathbf{a}_k.$$

The velocity vector $\dot{\mathbf{x}}$ of point $\mathbf{x}$ under any hyperbolic motion obeys

$$\dot{\mathbf{x}} = B\mathbf{x} \;\text{ with }\; B^T = -HBH.$$

$B := -H_1 + HA_1^T H$ is an example for such a 'pseudo-skew-symmetric' matrix. We superimpose this instantaneous motion and obtain the new velocity vectors

$$\mathbf{v}_{i,1} = \tfrac{1}{2}(A_1 + HA_1^T H)\mathbf{v}_i, \quad \mathbf{a}_{k,1} = -\tfrac{1}{2}(A_1 + HA_1^T H)\mathbf{a}_k. \tag{16}$$

In analogy to the Euclidean case these vectors in $\mathbb{M}^4$ are *perpendicular* in the Minkowski sense to the surface Φ_1. This follows from the fact that the plane tangent to Φ_1 at $\mathbf{x}$ obeys the equation $(HA_1 + A_1^T H)\mathbf{x} = 0$. The vector $(A_1 + HA_1^T H)\mathbf{x}$ is perpendicular to this tangent plane.

Since $\mathbf{v}_1,\dots,\mathbf{v}_4$ are supposed to be linearly independent, the velocity vectors of our flex are trivial if and only if the matrix $(A_1 + HA_1^T H)$ is pseudo-skew-symmetric. But then this must be the zero-matrix, and the bilinear form f_1 in (14) would be zero, too. Hence the existence of a second-order surface Φ_1 guarantees that the flex is non-trivial.

Second-order flexibility. The second order terms in the bilinear forms $f(t;\mathbf{x},\mathbf{y})$, $g(t;\mathbf{x},\mathbf{y})$ are listed in (15). Their difference

$$\begin{aligned} h_2(\mathbf{x},\mathbf{y}) :=\; & f_2(\mathbf{x},\mathbf{y}) - g_2(\mathbf{x},\mathbf{y}) = \\ =\; & \mathbf{x}^T(A_1^T HA_1 + A_1^{T2} H + HA_1HA_1^T H + HA_1^2)\mathbf{y} = \\ =\; & \mathbf{x}^T(A_1^T H + HA_1)H(HA_1 + A_1^T H)\mathbf{y} \end{aligned} \tag{17}$$

depends on the first order terms only. The points of the second-order surface $\Psi_2\colon h_2(\mathbf{x},\mathbf{x}) = 0$ have polar planes with respect to Φ_1 which are tangent to the absolute quadric Ω. This means for regular Φ_1 that Ψ_2 is polar to Ω with respect to Φ_1.

Due to (8), for a 2nd-order flexible octahedron $\mathcal{O}$ there is a surface $\Phi_{2b} : f_2(\mathbf{x},\mathbf{x}) = 0$ passing through the equator. The second surface $\Phi_{2a}\colon g_2(\mathbf{x},\mathbf{x}) = 0$ passes through the poles $\mathbf{a}_1$ and $\mathbf{a}_2$. And in addition, Φ_{2a} and Φ_{2b} span a pencil which includes Ψ_2. □

2. Flexible octahedra of Type 3 in $\mathbb{H}^3$

Any Bricard octahedron $\mathcal{O}$ of Type 3 in $\mathbb{E}^3$ admits flat positions which can be determined in the following way (see Fig. 1):

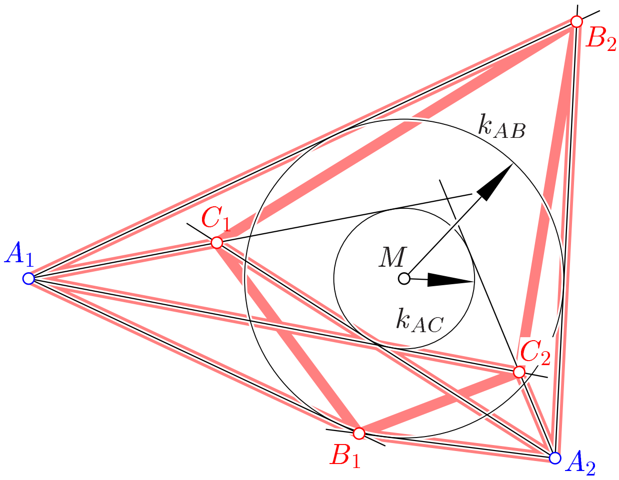

Figure 1. The flat position of Type 3, proper case

Let k_{AC}, k_{AB} be two different circles with the common center M, and let A_1, A_2 be two different points outside k_{AC} and k_{AB}. The tangent lines of k_{AB} passing through A_1 or A_2 define a quadrilateral. We specify (B_1, B_2) as a pair of opposite vertices.[3] Then $A_1B_1A_2B_2$ is a quadrangle with the four sides $A_1B_1, \ldots, B_2A_1$ tangent to k_{AB}. In the same way we specify a second quadrangle $A_1C_1A_2C_2$ tangent to the circle k_{AC}. Then (A_1, A_2), (B_1, B_2) and (C_1, C_2) are the pairs of opposite vertices of $\mathcal{O}$ in a flat position. We restrict to proper octahedra by the assumption that points $B_1, \ldots, C_2$ are finite. And we exclude self-symmetries by requiring that A_1, A_2 are not aligned with M and the distances $\overline{A_1M}$ and $\overline{A_2M}$ are different.

This definition can immediately be used in the hyperbolic plane $\mathbb{H}^2$, too. However, the demand for finite points $B_1, \ldots, C_2$ is of course much more restrictive. And we have to distinguish whether the concentric circles k_{AC}, k_{AB} are proper circles, hypercircles or horocircles in $\mathbb{H}^2$. This means that in the projective model which is used here the center M can be located in the interior, in the exterior or on the absolute conic u of $\mathbb{H}^2$.

[3]When k_{AB} happens to be tangent to the line A_1A_2 then the pair (B_1, B_2) is unique; one B-point is the point of contact.

We prove in several steps that in $\mathbb{H}^3$ an octahedron $\mathcal{O}$ with such a flat position is continuously flexible. Due to the projective model of $\mathbb{H}^3$ we can frequently follow the arguments given in [7] for the Euclidean case.

2.1 Properties of the flat positions

The pairs of line pencils with carriers (A_1, A_2), (B_1, B_2) and (C_1, C_2) span a two-parametric linear system $\mathcal{S}$ of second-class curves.

Any two different curves of this system span a one-parameter linear system ('range') which is completely included in $\mathcal{S}$. Therefore $\mathcal{S}$ contains the circles k_{AB} and k_{AC} and the spanned range, i.e., all circles centered at M, the multiplicity-two pencil with carrier M, and the absolute conic u, if seen as the set of isotropic lines.

Any two different ranges from $\mathcal{S}$ share one curve. This implies that also the quadrangle $B_1C_1B_2C_2$ is tangent to a circle k_{BC} centered at M. Hence at $\mathcal{O}$ no pair of opposite vertices can be distinguished among the others.

Furthermore, with any conic $c \in \mathcal{S}$ all conics confocal[4] to c are included in $\mathcal{S}$. And this range shares a curve c' with the range spanned by the pair of pencils with carriers (C_1, C_2) and the twofold pencil M. We therefore conclude for any position of M

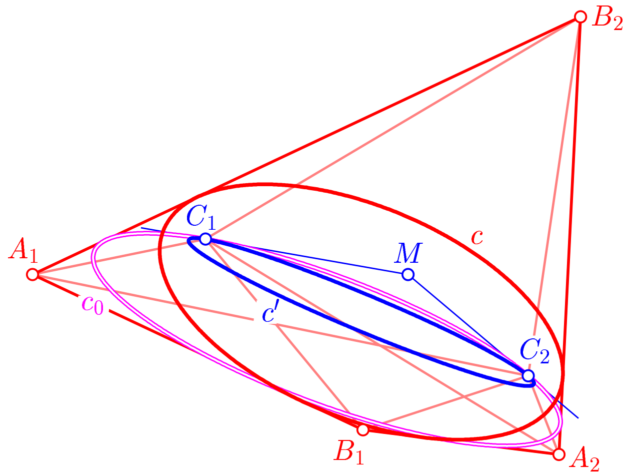

Figure 2. Illustration to Lemma 2

[4]Two conics in $\mathbb{H}^2$ are *confocal* if and only if these tangential conics span a linear system which contains the absolute conic u.

Lemma 2 *For any conic c tangent to the sides of $A_1B_1A_2B_2$ there is a confocal conic c' which passes simultanously through C_1 and C_2 such that the tangent lines at C_1 and C_2 intersect in M (Fig. 2).*

2.2 Particular case of Ivory's Theorem in $\mathbb{H}^3$

Let Φ' be a ruled quadric in $\mathbb{H}^3$ with a plane σ of symmetry. The principal section $\Phi' \cap \sigma$ is denoted by c'. Then the following Lemma 3 is a hyperbolic counterpart of Ivory's Theorem in $\mathbb{E}^3$. The affine transformation between 'corresponding points' of confocal surfaces in the Euclidean case is now replaced by a selfadjoint linear mapping l in the Weierstraß model or by a collinear transformation in the projective model of $\mathbb{H}^3$.

Lemma 3 *For any ruled quadric Φ' in $\mathbb{H}^3$ with a plane σ of symmetry and a real focal curve*[5] $c \subset \sigma$ *there is a selfadjoint linear mapping* $l\colon\ \mathbb{R}^4 \to \mathbb{R}^4$ *with the following properties:*

a) Points $\mathbf{x}' \in \Phi'$ are mapped on points $\mathbf{x} := l(\mathbf{x}') \in \sigma$ obeying $\langle\, \mathbf{x},\, \mathbf{x} \rangle = \langle\, \mathbf{x}',\, \mathbf{x}' \rangle$. Therefore absolute points of Φ' remain on Ω.

b) The restriction of l to $c' := \Phi' \cap \sigma$ is a bijection onto the focal curve c which must be located in the exterior of c'. Conversely, for any pair (c', c) of confocal curves in $\mathbb{H}^2$ with c in the exterior of c' there is a ruled surface Φ' through c' with c as focal conic.

c) The restriction of l to any generator of Φ' is an isometry onto a tangent line of c.

d) For any $\mathbf{x}', \mathbf{y}' \in \Phi'$ we obtain equal hyperbolic distances $d_h(l(\mathbf{x}'), \mathbf{y}') = d_h(\mathbf{x}', l(\mathbf{y}'))$.

Proof. Let $l : \mathbb{R}^4 \to \mathbb{R}^4$, $\mathbf{x}' \mapsto \mathbf{x} = l(\mathbf{x}')$ be selfadjoint with respect to the bilinear form $\langle\ ,\ \rangle$ in (1), i.e., $l = l^\star$ with $\langle\, l(\mathbf{x}'),\, \mathbf{y} \rangle = \langle\, \mathbf{x}',\, l^\star(\mathbf{y}) \rangle$. There is a quadric Φ' obeying

$$q(\mathbf{x}', \mathbf{x}') := \langle\, \mathbf{x}',\, \mathbf{x}' \rangle - \langle\, l(\mathbf{x}'),\, l(\mathbf{x}') \rangle = 0. \tag{18}$$

The points $\mathbf{x}' \in \Phi'$ are characterized by the property

$$\langle\, \mathbf{x}',\, \mathbf{x}' \rangle = \langle\, l(\mathbf{x}'),\, l(\mathbf{x}') \rangle = \langle\, \mathbf{x},\, \mathbf{x} \rangle.$$

[5]Two second-order surfaces Φ, Φ' in $\mathbb{H}^3$ are *confocal* if and only if their dual surfaces span a linear system which includes the absolute surface Ω — seen as the set of absolute planes. If the dual of Φ is singular, i.e., consisting of the tangential planes of a conic c, then c is called a *focal conic* of Φ'.

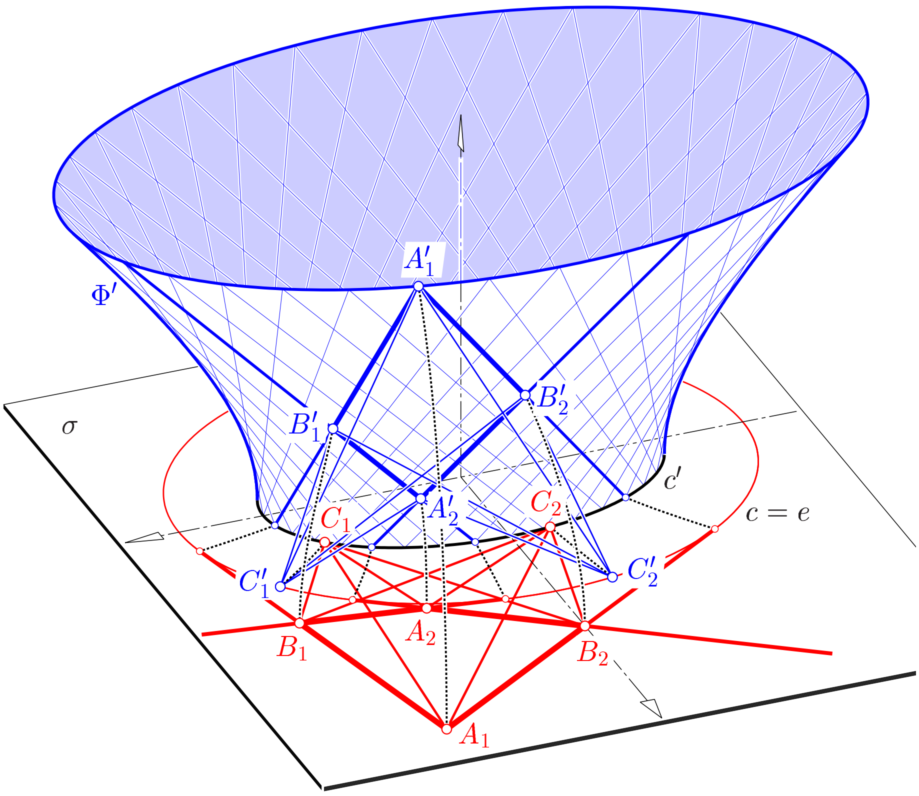

Figure 3. Proving the flexibility of Type 3 with Ivory's Theorem

Therefore according to (2) for any two points $\mathbf{x}', \mathbf{y}' \in \Phi'$ we have a statement of Ivory type

$$d_h(l(\mathbf{x}'), \mathbf{y}) = d_h(\mathbf{x}', l^\star(\mathbf{y})).$$

In addition, the distance between two points $\mathbf{x}', \mathbf{y}' \in \Phi'$ is preserved if and only if

$$q(\mathbf{x}', \mathbf{y}') = \langle\, \mathbf{x}',\, \mathbf{y}' \rangle - \langle\, l(\mathbf{x}'),\, l(\mathbf{y}') \rangle = 0.$$

This characterizes conjugate position of $\mathbf{x}', \mathbf{y}'$ with respect to Φ', i.e., the connecting line $\mathbf{x}'\mathbf{y}'$ is a generator of Φ'.

Suppose the selfadjoint l has rank 3 with $l(\mathbb{R}^4) = \sigma$. Let $\mathbf{s}$ denote the absolute pole of σ. Then we have

$$0 = \langle\, \mathbf{s},\, l(\mathbf{x}') \rangle = \langle\, l(\mathbf{s}),\, \mathbf{x}' \rangle$$

for all $\mathbf{x}' \in \mathbb{R}^4$. This implies $l(\mathbf{s}) = \mathbf{o}$, i.e., the fibres of l are perpendicular to σ.

σ needs to be a plane of symmetry for Φ' since for all $\mathbf{y}' \in \sigma$ we have $\langle\, \mathbf{y}', \mathbf{s}\rangle = 0$ and therefore according to (18)

$$q(\mathbf{y}', \mathbf{s}) := \langle\, \mathbf{y}', \mathbf{s}\rangle - \langle\, l(\mathbf{y}'),\, l(\mathbf{s})\rangle = 0.$$

This means, $\mathbf{s}$ is the pole of σ also with respect to Φ'. The restriction of l to the plane σ is bijective and transfers $c' = \Phi' \cap \sigma$ onto a conic c. All points of $\Phi' \setminus c'$ are mapped into the exterior of c since the images of the generators of Φ' are lines which meet c in exactly one point, i.e., tangent lines.

We prove by contradiction that c' must be located in the interior of c: Suppose there is a point $\mathbf{x}' \in c'$ which coincides with the image $\mathbf{y}$ of any $\mathbf{y}' \in \Phi' \setminus c'$. Then Ivory's Theorem would give $0 = d_h(\mathbf{x}', \mathbf{y}) = d_h(\mathbf{x}, \mathbf{y}')$ with $\mathbf{x} \in c \subset \sigma$ and $\mathbf{y}'$ distant from σ, and this is a contradiction.

We continue the proof of Lemma 3 by showing that for each ruled quadric Φ' there is a selfadjoint mapping l of rank 3 such that the equation of Φ' is of the form (18). And we show that Φ' is confocal to its image under l. For this purpose we turn over to matrix notation:

Without restriction of generality we suppose that σ obeys the equation $x_3 = 0$. Then we can set up the selfadjoint mapping as

$$l\colon\ \mathbf{x}' \mapsto \mathbf{x} = A\mathbf{x}' \quad \text{with} \quad A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & 0 \\ -a_{01} & a_{11} & a_{12} & 0 \\ -a_{02} & a_{12} & a_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

since $A^T = HAH$. Hence equation (18) gets the form

$$\Phi'\colon\ q(\mathbf{x}', \mathbf{x}') = \mathbf{x}'^T (H - A^T H A)\mathbf{x}' = \mathbf{x}'^T H (I_4 - A^2)\mathbf{x}' = 0. \tag{20}$$

The symmetric coordinate matrix

$$Q := H - A^T H A$$

of the quadratic form $q(\mathbf{x}', \mathbf{x}')$ is supposed to be regular, and from (20) we obtain

$$A^2 = I_4 - HQ. \tag{21}$$

Points $\mathbf{x}'$ of any plane with coordinate vector $\mathbf{u}' = A^T\mathbf{u}$ have their image $\mathbf{x} = l(\mathbf{x}')$ in the plane $\mathbf{u}^T\mathbf{x} = 0$. The mapping $\mathbf{u} \mapsto A^T\mathbf{u}$ is adjoint to l with respect to the standard scalar product. Under this mapping all planes obeying the quadratic form

$$\Phi\colon\ \mathbf{u}^T A (H - A^T H A)^{-1} A^T \mathbf{u} = 0 \tag{22}$$

are transferred into planes tangent to Φ'. So, (22) is the dual representation of the image $l(\Phi')$, while $\mathbf{u}'^T(H - A^T HA)^{-1}\mathbf{u}' = 0$ is the dual equation of Φ'.

Now we state

$$H + A(H - A^T HA)^{-1}A^T = (H - A^T HA)^{-1}. \tag{23}$$

This reveals that $l(\Phi')$ and the dual of the absolute quadric Ω with coordinate matrix H span a range which includes Φ', i.e., Φ' and $l(\Phi')$ are confocal — whether l has rank 4 or 3.

For proving eq. (23) we multiply both sides with $(H - A^T HA)$ and obtain

$$I_4 - HA^T HA + A(H - A^T HA)^{-1}A^T(H - A^T HA) = I_4,$$

and this is an identity because of $HA^T = AH$ and

$$A^T(H - A^T HA) = A^T H - A^T A^T HA = (H - A^T HA)A.$$

It remains to prove that each hyperboloid Φ' in $\mathbb{H}^3$ with σ as a plane of symmetry has an equation of type (20). This is equivalent to the existence of a matrix A, which is of type (19) and obeys (21).

There are three types of ruled quadrics Φ' to distinguish: The principal section $c' = \Phi' \cap \sigma$ has

(I) a center of symmetry,

(II) no center, but an axis of symmetry,

(III) neither a center, nor an axis of symmetry.

Ad (I): In this case we can set up the equation of Φ' as

$$q(\mathbf{x}', \mathbf{x}') = \gamma_{00}{x'_0}^2 + \gamma_{11}{x'_1}^2 + \gamma_{22}{x'_2}^2 - {x'_3}^2 = 0.$$

This is a ruled quadric with points in the interior of the absolute Ω and with a real focal conic c in σ if either

$$\gamma_{00} < 0, \quad \gamma_{11}, \gamma_{22} > 0, \quad \gamma_{00} + \gamma_{11} > 0 \;\text{ or }\; \gamma_{00} + \gamma_{11} > 0,$$

or

$$0 < \gamma_{00} < 1, \quad -1 < \gamma_{11} < 0, \quad \gamma_{22} > 0, \quad \gamma_{00} + \gamma_{11} > 0.$$

In both cases there is a matrix A obeying (21), namely

$$A^2 = \mathrm{diag}(1 - \gamma_{00},\; 1 + \gamma_{11},\; 1 + \gamma_{22},\; 0).$$

We get the solutions

$$A = \operatorname{diag}(\pm\sqrt{1-\gamma_{00}},\ \pm\sqrt{1+\gamma_{11}},\ \pm\sqrt{1+\gamma_{22}},\ 0),$$

and the corresponding focal curve c obeys

$$x_3 = 0 \ \text{ and } \ \frac{\gamma_{00}}{1-\gamma_{00}}\,x_0^2 + \frac{\gamma_{11}}{1+\gamma_{11}}\,x_1^2 + \frac{\gamma_{22}}{1+\gamma_{22}}\,x_0^2 = 0.$$

Ad (II): Without restriction of generality we can set up

$$q(\mathbf{x}',\mathbf{x}') = \gamma_{11}{x_1'}^2 + 2\gamma_{01}x_0'x_1' + \gamma_{22}{x_2'}^2 - {x_3'}^2 = 0$$

with

$$\gamma_{01} \neq 0, \quad \gamma_{11}^2 \leq 4\gamma_{01}^2, \quad \gamma_{22} > 0.$$

According to (21) there is a matrix A obeying

$$A^2 = \begin{pmatrix} 1 & -\gamma_{01} & 0 & 0 \\ \gamma_{01} & 1+\gamma_{11} & 0 & 0 \\ 0 & 0 & 1+\gamma_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

since the upper-left 2×2-matrix has either conjugate complex eigenvalues or a twofold eigenvalue with a one-dimensional eigenspace. We obtain

$$A = \begin{pmatrix} (-\delta+\gamma_{11})/2\sqrt{\delta} & \gamma_{01}/\sqrt{\delta} & 0 & 0 \\ -\gamma_{01}/\sqrt{\delta} & -(\delta+\gamma_{11})/2\sqrt{\delta} & 0 & 0 \\ 0 & 0 & \sqrt{1+\gamma_{22}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for $\delta := 2 + \gamma_{11} + 2\sqrt{1+\gamma_{01}^2+\gamma_{11}}$. The focal curve $c \in \sigma$ obeys

$$(\gamma_{22}+1)\left[\gamma_{01}^2 x_0^2 - 2\gamma_{01}x_0x_1 - (\gamma_{01}^2+\gamma_{11})x_1^2\right] - \gamma_{22}(1+\gamma_{01}^2+\gamma_{11})x_2^2 = 0.$$

Ad (III): We can set up

$$q(\mathbf{x}',\mathbf{x}') = \gamma_{00}({x_0'}^2 - {x_1'}^2 - {x_2'}^2) + 2\gamma_{02}(x_0' - x_1')x_2' - {x_3'}^2 = 0$$

with

$$\gamma_{00} < 0, \quad \gamma_{02} \neq 0.$$

There is a matrix A obeying

$$A^2 = \begin{pmatrix} 1-\gamma_{00} & 0 & -\gamma_{02} & 0 \\ 0 & 1-\gamma_{00} & -\gamma_{02} & 0 \\ \gamma_{02} & -\gamma_{02} & 1-\gamma_{00} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

since the upper-left 3×3-matrix has a positive triple eigenvalue with a one-dimensional eigenspace. The solution is

$$A = \frac{1}{8\sqrt{\delta^3}} \begin{pmatrix} 8\delta^2 + \gamma_{02}^2 & -\gamma_{02}^2 & -4\gamma_{02}\delta & 0 \\ \gamma_{02}^2 & 8\delta^2 - \gamma_{02}^2 & -4\gamma_{02}\delta & 0 \\ 4\gamma_{02}\delta & -4\gamma_{02}\delta & 8\delta^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for $\delta := 1 - \gamma_{00}$; the focal curve $c \in \sigma$ obeys

$$\gamma_{00}\delta^2(x_0^2 - x_1^2 - x_2^2) - \gamma_{02}(x_0 - x_1)\,[\gamma_{02}(x_0 - x_1) - 2\delta x_2] = 0.$$

The last remark of Lemma 3,b can be verified by comparing the equations of c with that of Φ' in the cases (I)–(III). □

2.3 Conclusions for Octahedra in $\mathbb{H}^3$

Now we combine the previous statements: We identify σ with the projective model of the hyperbolic plane $\mathbb{H}^2$ where the flat position of $\mathcal{O}$ is located. We see each conic c' of Lemma 2 (Fig. 2) as principal section of a one-sheet hyperboloid Φ' and c as its focal curve (see Fig. 3).

Then Lemma 3,c reveals that there is a quadrangle $A_1'B_1'A_2'B_2'$ with sides on Φ' which is mapped by l onto $A_1B_1A_2B_2$ while all side-lengths are preserved.[6] Under l the vertices $C_1, C_2 \in c'$ are mapped onto $C_1', C_2' \in c$ (notation reversed!), and Ivory's Theorem in Lemma 3,d implies $d_h(A_i, C_j) = d_h(A_i', C_j')$ and $d_h(B_i, C_j) = d_h(B_i', C_j')$ (see Fig. 3). Hence the spatial octahedron $A_1' \dots C_2'$ is isometric to the flat position $A_1 \dots C_2$.

For completing the proof of the continuous flexibility of $\mathcal{O}$ two items remain to be checked:

(i) c' needs to be inside the focal curve c, to say, no tangent line of c may intersect c', and

(ii) c and c' must be of the same type with respect to u, i.e., both intersect u in the same way.

It is substantial that due to the properties of the linear system $\mathcal{S}$ there is a conic c_0 tangent to $A_1B_1A_2B_2$ and passing through both line elements (C_i, MC_i), $i = 1, 2$. So we can use continuity arguments:

Ad (i): Let t denote the side A_1B_1. While the 2nd-class curve c' with line elements (C_i, MC_i) varies, the pole T of t with respect to c' traces

[6] The quadrangle $A_1'B_1'A_2'B_2' \in \Phi$ is unique up to the reflection in the plane σ.

a line t'. For $c' = c_0$ we obtain the point T_0 of contact between t and c_0. We get $T = M$ when c degenerates into the pencil with carrier M. And $S = t' \cap C_1C_2$ is the pole of t with respect to the pair of line pencils (C_1, C_2). Conversely, any point T of line T_0M defines a unique curve c' of this contact range.

Now it depends on the choice of direction when starting from c_0: If T moves along t' torwards the interior of c_0, i.e., if the pair (T, M) separates (T_0, S), then the corresponding conic c' will not intersect t. This results from properties of the polarity with respect to c' and the involution of conjugate points on t'. So c' meets the necessary condition; it is included in the interior of the confocal c, which according to Lemma 2 is tangent to t.

Ad (ii): When starting from c_0, the types of c and c' with respect to the absolute conic u can only begin to differ at a position where c or c' contacts u. Since c and c' are confocal, this contact with u happens for both conics simultaneously at the same point U. So, it could only happen that — from this contact at U on — one conic has real points of intersection near U, the other has no intersection. But this is a contradiction with Lemma 3,a,b, which states that there is a bijection $c' \to c$ mapping absolute points again on absolute points, provided c' is in the interior of c.

All octahedra of Type 3 admit a *second flat position*. This results from the concentric circles k_{AB}, k_{AC}, k_{BC} in the given flat position (see Fig. 1) for the following reason:

At each of the six vertices, e.g. at A_i, the connecting lines with the other pairs (B_1, B_2) and (C_1, C_2) are symmetric with respect to the line through M: Suppose we keep the face $A_1B_1C_1$ fixed. Then for the second flat position it is necessary that each vertex $\overline{A}_2, \overline{B}_2, \overline{C}_2$ of the opposite face is obtained by reflecting the single points A_2, B_2 and C_2 in the sides B_1C_1, A_1C_1 and A_1B_1, respectively (see Fig. 4). In order to guarantee that the distances do not change, we must e.g. demonstrate that there is one isometry in $\mathbb{H}^2$ which maps simultaneously $B_2 \mapsto \overline{B}_2$ and $C_2 \mapsto \overline{C}_2$. The first can be carried out by the consecutive reflections in the lines A_1B_2 and A_1C_1. For the latter we use the reflections in A_1C_2 and A_1B_1. Now it results from the Three-Reflection-Theorem of absolute geometry that these products of reflections are equal because of the symmetry with respect to line A_1M.

It turns out that in the sense of Fig. 3 this second flat position is reached when c' degenerates into the pair of line pencils (C_1, C_2). The corresponding hyperboloid Φ degenerates into a focal conic of c_0.

Thus we end up with

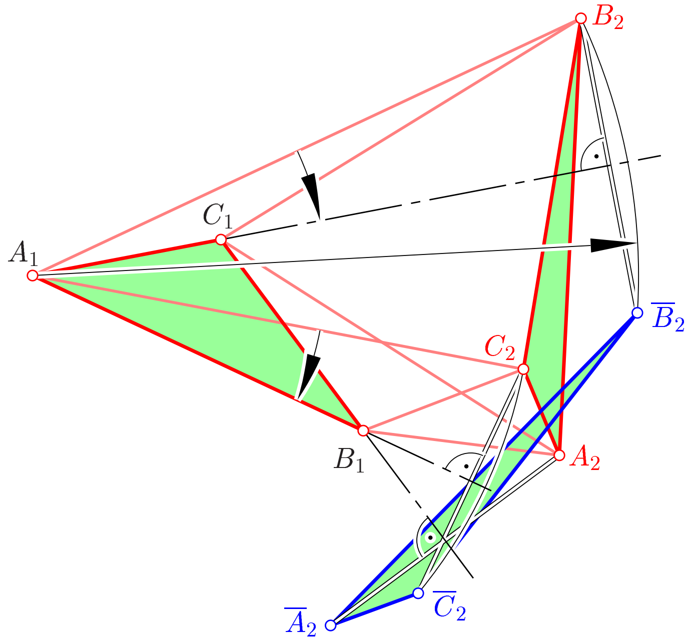

Figure 4. The two flat positions $A_1B_1C_1A_2B_2C_2$ and $A_1B_1C_1\overline{A}_2\overline{B}_2\overline{C}_2$ of $\mathcal{O}$

Theorem 2 *All three classes of Type 3 octahedra in* $\mathbb{H}^3$ *are continuously flexible and they admit a second flat position.*

Theorem 3 *There are at least three types of continuously flexible octahedra in* $\mathbb{H}^3$. *At Type 1 all pairs of opposite vertices are symmetric with respect to a line, at Type 2 two pairs of vertices are symmetric with respect to a plane which passes through the remaining two vertices. Flexible octahedra of Type 3 are unsymmetric with flat positions according to Fig. 1.*

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