Jordan Systems and Associated Geometric Structures

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Outline of the talk

- Introduction: Jordan systems, chain geometries, and their connections (an overview)
- Jordan systems and related algebraic structures
- Chain geometries and their subspaces
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Introduction: Jordan systems

K commutative field

R associative K-algebra (with 1), i.e. a ring with $K \subseteq Z(R)$, $1_K = 1_R =: 1$

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$J \subseteq R$ is called a **Jordan system** in R, if:

- J is a subspace of the vector space R over K,
- $1 \in J$,
- $a \in J$, a invertible in $R \Rightarrow a^{-1} \in J$ (i.e., J is closed under inversion)

Introduction: Jordan systems

Example: $R = \mathrm{M}(2 \times 2, K)$ matrix algebra

$J = \{\text{symmetric matrices}\} = \{A \in R \mid A = A^t\}$

$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in J$ invertible $\Longrightarrow A^{-1} = \frac{1}{ac-b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \in J$

Introduction: chain geometry

The **chain geometry** associated to a K-algebra R:

$\Sigma = \Sigma(K, R) = (\mathbb{P}, \mathcal{C})$ (points, "chains")

where, in particular, $\mathbb{P} = \mathbb{P}(R)$ is the projective line over the ring R

Abstract (synthetic) concept: **chain space**

Introduction: chain geometry

Example: $\Sigma(\mathbb{R}, \mathbb{C})$, the **real Möbius plane**:

$\mathbb{P}$: points on a sphere in $\mathbb{R}^3$

$\mathcal{C}$: circles on the sphere $=$ plane sections of the sphere

Introduction: chain geometry

Example: $\Sigma(\mathbb{R}, \mathbb{C})$, the **real Möbius plane**:

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$\mathcal{C}$: circles on the sphere $=$ plane sections of the sphere

Stereographic projection from the north pole $n \in \mathbb{P}$:

$\mathbb{P} \to \mathbb{C} \cup \{\infty\} = \mathbb{P}(\mathbb{C})$,

and each circle $C \in \mathcal{C}$ is mapped to a circle in $\mathbb{C}$ or (if $n \in C$) to an extended line $L \cup \{\infty\}$

Introduction: chain geometry

$\Sigma = (\mathbb{P}, \mathcal{C})$ chain space

S is a **subspace** of Σ, if

- $S \subseteq \mathbb{P}$,
- $(S, \mathcal{C}(S))$ is a chain space, where $\mathcal{C}(S) = \{$all chains contained in $S\}$.

Introduction: chain geometry

Example:

Let $\mathcal{Q}$ be quadric in $\mathrm{PG}(n, K)$, $n > 3$ (with certain properties).

Then, using the plane sections of $\mathcal{Q}$, one obtains a chain space $\Sigma(\mathcal{Q})$.

Let U be a projective subspace of $\mathrm{PG}(n, K)$.

Then $\mathcal{Q}' = \mathcal{Q} \cap U$ is a quadric in U and a subspace of $\Sigma(\mathcal{Q})$.

Introduction: chain geometry

Theorem. (A. Herzer 1992). Under certain conditions:

subspaces of $\Sigma(K, R) \longleftrightarrow$ Jordan systems in R.

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- **Jordan systems and related algebraic structures**
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Jordan systems and related algebraic structures

Jordan systems:

- named after Pascual Jordan (1902-1980), German physicist
- name due to Herzer
- connections to: Jordan algebras, Jordan homomorphisms, . . .

Jordan systems and related algebraic structures

Special Lie and Jordan algebras

R associative K-algebra

$\longrightarrow$ Lie algebra $R^- = (R, +, [\ ,\ ])$, where $[a, b] = ab - ba$

A Lie subalgebra of some R^- is called a special Lie algebra; and one can show that every Lie algebra is special.

Jordan systems and related algebraic structures

Special Lie and Jordan algebras

R associative K-algebra

$\longrightarrow$ Lie algebra $R^- = (R, +, [\ ,\ ])$, where $[a, b] = ab - ba$

A Lie subalgebra of some R^- is called a special Lie algebra; and one can show that every Lie algebra is special.

$\longrightarrow$ Jordan algebra $R^+ = (R, +, \circ)$, where $a \circ b = \frac{1}{2}(ab + ba)$ $(\mathrm{char} K \neq 2)$

A Jordan subalgebra of some R^+ is called a special Jordan algebra; and **not** every Jordan algebra is special.

Jordan systems and related algebraic structures

A (commutative) **Jordan algebra** is a (non-associative) K-algebra $(R, +, \circ)$ satisfying

- $a \circ b = b \circ a$ (commutativity)
- $(a \circ b) \circ (a \circ a) = a \circ (b \circ (a \circ a))$ (Jordan identity)

Example: Let M be the set of all matrices of the following type:

$$\begin{pmatrix} \alpha & x & y \\ \bar{x} & \beta & z \\ \bar{y} & \bar{z} & \gamma \end{pmatrix}, \ \alpha, \beta, \gamma \in \mathbb{R}, x, y, z \in \mathbb{O}$$

Then $(M, +, \circ)$ with $A \circ B = \frac{1}{2}(AB + BA)$ is an exceptional (i.e. not special) Jordan algebra.

Jordan systems and related algebraic structures

Let R be an associative K-algebra, let R^* the set of units (multiplicatively invertible elements) of R. Let $J \subseteq R$ be a subspace of the vector space R with $1 \in J$. Then we call J

- **Jordan system** in R, if $\forall a \in J \cap R^* : a^{-1} \in J$.
- **Jordan closed** in R, if $\forall a, b \in J : aba \in J$.
- **strong** in R, if $\forall a \in J$: $|e(a)| > |K \setminus e(a)|$, where $e(a) = \{k \in K \mid k + a \in R^*\}$.

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Proposition. (Herzer). Let J be a strong Jordan system in R. Then J is Jordan closed in R.

Jordan systems and related algebraic structures

Let J be Jordan closed in R (e.g. J a strong Jordan system in R). Then

- J is closed with respect to squaring: For $a \in J$ we have $a^2 = a \cdot 1 \cdot a \in J$.
- For $a, b \in J$ also $ab + ba \in J$, since $ab + ba = (a + b)^2 - a^2 - b^2$.

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So in case of $\mathrm{char} K \neq 2$ we have that J is a special Jordan algebra (a Jordan subalgebra of R^+).

Jordan systems and related algebraic structures

Example 1: $R = K[t]$ polynomial ring; then $R^* = K^* = K \setminus \{0\}$.

$J = K + Kt$ is a subspace of the vector space R with $1 \in J$.

- J is not closed w.r.t. multiplication,
- J is a Jordan system in R:

  $a = \alpha + \beta t \in J \cap R^* \implies \alpha \neq 0$ and $\beta = 0 \implies a^{-1} = \alpha^{-1} \in J$.
- J is not Jordan closed in R: $t \cdot 1 \cdot t = t^2 \notin J$.
- J is not strong in R: $e(t) = \emptyset$.

Example 2: $R = K[t]/(t^3)$ (chain ring); then $R^* = R \setminus Kt + Kt^2$.

$J = K + Kt$ is a subspace of the vector space R with $1 \in J$.

- J is not closed w.r.t. multiplication,
- J is not a Jordan system in R:

  $a = 1 + t \in J \cap R^*$ but $a^{-1} = 1 - t + t^2 \notin J$.
- J is not Jordan closed in R: $t \cdot 1 \cdot t = t^2 \notin J$.
- J is strong in R (if $|K| > 2$):

  $e(\alpha + \beta t) = \{k \in K \mid k + \alpha + \beta t \in R^*\} = \{k \in K \mid k \neq -\alpha\}$.

Jordan systems and related algebraic structures

Example 3: $R = \mathrm{M}(n \times n, K)$ matrix algebra

$J = \{A \in R \mid A = A^t\}$ (symmetric matrices)

J is a Jordan system in R and also Jordan closed in R:

$(A^{-1})^t = (A^t)^{-1}$, $\quad (ABA)^t = A^t B^t A^t$.

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Generalization: R an arbitrary K-algebra, κ an anti-automorphism of R (i.e., $(ab)^\kappa = b^\kappa a^\kappa$). Then $J = \mathrm{Fix}\kappa = \{a \in R \mid a = a^\kappa\}$ is a Jordan closed Jordan system in R.

Jordan systems and related algebraic structures

Example 4: $\mathbb{O}$ Cayley's octonions (8-dimensional real non-associative division algebra); $R = \text{End}_{\mathbb{R}}(\mathbb{O})$ endomorphism ring of the vector space ${}_{\mathbb{R}}\mathbb{O}$ (so $R \cong \mathrm{M}(8 \times 8, \mathbb{R})$).

$J = \{\rho_u : x \mapsto xu \mid u \in \mathbb{O}\}$ (right multiplications)

J is a Jordan system in R and also Jordan closed in R, because in $\mathbb{O}$ the following identities are valid:

- $(xu)u^{-1} = x$ $\quad$ $(\Longrightarrow (\rho_u)^{-1} = \rho_{u^{-1}})$
- $((xu)v)u = x(uvu)$ $\quad$ $(\Longrightarrow \rho_u \rho_v \rho_u = \rho_{uvu})$

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Generalization: The same construction works for any algebra (instead of $\mathbb{O}$) that satisfies the two identities above.

Jordan systems and related algebraic structures

A **Jordan pair** is a pair (V^+, V^-) of vector spaces over K with two trilinear maps $T_\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm$ satisfying

- $T_\pm(x, a, z) = T_\pm(z, a, x)$
- $T_\pm(x, a, T_\pm(y, b, z)) - T_\pm(y, b, T_\pm(x, a, z))$
  $= T_\pm(T_\pm(x, a, y), b, z) + T_\pm(y, T_\mp(a, x, b), z)$

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Each Jordan algebra $(A, +, \circ)$ gives rise to a Jordan pair as follows: $V^+ = V^- = A$, $T_\pm(a, b, c) = (a \circ b) \circ c + (b \circ c) \circ a - (a \circ c) \circ b$

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Example: $V^+ = \mathrm{M}(n \times m, K)$, $V^- = \mathrm{M}(m \times n, K)$,
$T_\pm(A, B, C) = ABC + CBA$.

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W. Bertram (2002) associated generalized projective geometries to Jordan pairs.

Jordan systems and related algebraic structures

Let J_1, J_2 be strong Jordan systems in K-algebras R_1, R_2.

A pair (α, β) of K-semilinear mappings $J_1 \to J_2$ is called an homotopism, if

- $1^\alpha \in J_2^* = J_2 \cap R_2^*$,
- $\forall a, b \in J_1 : (aba)^\alpha = a^\alpha b^\beta a^\alpha$.

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If $1^\alpha = 1$, then $\alpha = \beta$: $x^\alpha = (1x1)^\alpha = 1^\alpha x^\beta 1^\alpha = x^\beta$.
Such an α is called a Jordan homomorphism.

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Such an α is called a Jordan homomorphism.

If $R_1 = R_2$, then for $u \in J_1^*$ the pair (α, β) with $\alpha : x \mapsto ux$, $\beta : x \mapsto xu^{-1}$ is an isotopism $J_1 \to J_2$, where $J_2 = uJ_1 (= J_1 u^{-1})$. We call it principal isotopism.

Jordan systems and related algebraic structures

In particular, if J is a strong Jordan system in R and $u \in J^*$, then also $J' = uJ$ is a strong Jordan system in R (isotopic to J).

Example:

$J' = \left\{ \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \mid a, b, c \in K \right\}$ is isotopic to the Jordan system of

symmetric matrices via $X \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X$

Jordan systems and related algebraic structures

Examples of Jordan endomorphisms:

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Jordan systems and related algebraic structures

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2) J a ring (i.e. closed w.r.t. multiplication): each ring endomorphism or anti-endomorphism is a Jordan endomorphism

3) $J = R_1 \times R_2$ direct product of rings, α_1 endomorphism of R_1, α_2 anti-endomorphism of R_2. Then $\alpha : J \to J : (x_1, x_2) \mapsto (x_1^{\alpha_1}, x_2^{\alpha_2})$ is a (proper) Jordan homomorphism.

Jordan systems and related algebraic structures

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4) $J = \{\rho_u : x \mapsto xu \mid u \in \mathbb{O}\}$ (right multiplications in the octonions), $\rho_c \in J^*$ fixed (i.e. $c \in \mathbb{O}^*$). Then $\alpha : \rho_u \mapsto (\rho_c)^{-1}\rho_u\rho_c\,(= \rho_{u^{-1}cu})$, is a Jordan homomorphism, due to Moufang's identities.

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Chain geometries and their subspaces

The **chain geometry** over a K-algbra R:

$\Sigma(K,R) = (\mathbb{P}(R), \mathcal{C}(K,R))$, where

$\mathbb{P}(R) = \{R(1,0)M \mid M \in \mathrm{GL}(2,R)\}$ **projective line** over R

$= \{R(a,b) \mid (a,b) \in R^2$ first row of an invertible matrix$\}$

Chain geometries and their subspaces

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$\mathcal{C}(K,R) = \{C_0 M \mid M \in \mathrm{GL}(2,R)\}$, where

$C_0 = \{R(1,0)N \mid N \in \mathrm{GL}(2,K)\} = \{R(k,1) \mid k \in K\} \cup \{R(1,0)\}$,

so the chains are the K-sublines of $\mathbb{P}(R)$.

Chain geometries and their subspaces

In other words:

The point set $\mathbb{P}(R)$ arises from the standard point $p_0 = R(1,0)$ by taking all its images under the action of $\mathrm{GL}(2,R)$:

$$R(x,y) \mapsto R(x,y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = R(xa+yc, xb+yd)$$

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$$R(x,y) \mapsto R(x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = R(xa+yc, xb+yd)$$

The chain set arises in the same way from the standard chain

$$C_0 = \{R(k,1) \mid k \in K\} \cup \{R(1,0)\}$$

(which can be considered as the projective line over K).

Chain geometries and their subspaces

$\Sigma(K,R)$ satisfies the axioms of a **chain space** $\Sigma = (\mathbb{P}, \mathcal{C})$:

(CS1) Every chain contains at least three points, every point lies on at least one chain.

(CS2) Any three pairwise distant points lie together in exactly one chain.

Here two points are called distant, if they are different and joined by at least one chain.

(CS3) For every point p the residual space $\Sigma_p = (D(p), \mathcal{C}(p))$, with $D(p) = \{q \in \mathbb{P} \mid q$ distant to $p\}$, $\mathcal{C}(p) = \{C \setminus \{p\} \mid p \in C \in \mathcal{C}\}$, is a partial affine space, i.e. an affine space with some parallel classes of lines missing.

Chain geometries and their subspaces

Quadric chain spaces:

Let $\mathcal{Q}$ be a quadric in $\mathrm{PG}(n, K)$ satisfying the following conditions:

- $\mathcal{Q}$ possesses a secant
- $\mathcal{Q}$ is not contained in the union of two hyperplanes

Then $\Sigma(\mathcal{Q}) = (\mathbb{P}(\mathcal{Q}), \mathcal{C}(\mathcal{Q}))$ defined below is a chain space:

$\mathbb{P}(\mathcal{Q}) = \{p \in \mathcal{Q} \mid p$ not a double point$\}$, where a point p is a double point if the tangent space at p is the whole projective space,

$\mathcal{C}(\mathcal{Q}) = \{\mathcal{Q} \cap E \mid E$ admissible plane$\}$, where a plane E is called admissible, if $\mathcal{Q} \cap E$ contains at least three points but no line (so the chains are oval conics).

Chain geometries and their subspaces

Examples of quadric chain spaces:

1) $\mathcal{Q}$ a quadratic cone in $\mathrm{PG}(3,\mathbb{R})$: Then $\Sigma(\mathcal{Q})$ is the real Laguerre plane, isomorphic to $\Sigma(\mathbb{R},\mathbb{D})$, where $\mathbb{D}$ is the ring of dual numbers over $\mathbb{R}$, i.e. $\mathbb{D} = \mathbb{R} + \mathbb{R}\varepsilon$ with $\varepsilon^2 = 0$.

Chain geometries and their subspaces

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2) $\mathcal{Q}$ a hyperbolic quadric in $\mathrm{PG}(3,\mathbb{R})$: Then $\Sigma(\mathcal{Q})$ is the real Minkowski plane, isomorphic to $\Sigma(\mathbb{R},\mathbb{A})$, where $\mathbb{A}$ is the ring of double numbers over $\mathbb{R}$, i.e. $\mathbb{A} = \mathbb{R} \times \mathbb{R}$ (direct product).

Chain geometries and their subspaces

Examples of quadric chain spaces:

3) $\mathcal{Q}$ the **Klein quadric** in $\mathrm{PG}(5, K)$. Then the Klein correspondence yields that $\Sigma(\mathcal{Q})$ is isomorphic to the geometry $\Sigma' = (\mathbb{P}', \mathcal{C}')$, where

$\mathbb{P}' = \{$all lines in $\mathrm{PG}(3, K)\}$, $\mathcal{C}' = \{$all reguli in $\mathrm{PG}(3, K)\}$.

Chain geometries and their subspaces

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Moreover, $\Sigma(\mathcal{Q})$ is isomorphic to the chain geometry $\Sigma(K,R)$, where $R = \mathrm{M}(2 \times 2, K)$. The mapping

$$R(A,B) \mapsto \text{row space } (A\ B)$$

is an isomorphism $\Sigma(K,R) \to \Sigma'$.
(Note that $R(A,B) \in \mathbb{P}(R) \Leftrightarrow \mathrm{rk}(A\ B) = 2$.)

Chain geometries and their subspaces

Let $\Sigma = (\mathbb{P}, \mathcal{C})$ be a chain space.

A subset $S \subseteq \mathbb{P}$ is called a **subspace** of Σ, if $(S, \mathcal{C}(S))$ is a chain space, where $\mathcal{C}(S) = \{$all chains contained in $S\}$.

Equivalently, S satisfies the following conditions:

- If $p, q, r \in S$ are pairwise distant, then the (unique) chain through p, q, r is contained in S.
- If $p, q \in S$ are distant and $C \in \mathcal{C}$ contains q, then the unique chain C' through p contacting C in q is contained in S.

Chain geometries and their subspaces

Examples: Subspaces of $\Sigma(\mathcal{Q})$, where $\mathcal{Q}$ is the Klein quadric:

Let U be 3-dimensional projective subspace of $\mathrm{PG}(5, K)$. Then $\mathcal{Q}' = \mathcal{Q} \cap U$ gives rise to a subspace of $\Sigma(\mathcal{Q})$.

There are three types: The line $U^{\perp}$ is either a tangent, a secant, or an external line.

Chain geometries and their subspaces

Type 1: $U^\perp$ tangent: Then $\mathcal{Q}'$ is a cone and $\Sigma(\mathcal{Q}')$ is a Laguerre plane.

The associated algebra $K(\varepsilon)$ of dual numbers over K can be found as a subalgebra in $R = \mathrm{M}(2 \times 2, K)$ via

$$a + b\varepsilon \longmapsto \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

The corresponding line model in $\mathrm{PG}(3, K)$ is a parabolic linear congruence.

Chain geometries and their subspaces

Type 2: $U^\perp$ secant: Then $\mathcal{Q}'$ is a hyperbolic quadric and $\Sigma(\mathcal{Q}')$ is a Minkowski plane.

The associated algebra $K \times K$ of double numbers over K can be found as a subalgebra in $R = \mathrm{M}(2 \times 2, K)$ via

$$(a, b) \longmapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

The corresponding line model in $\mathrm{PG}(3, K)$ is a hyperbolic linear congruence.

Chain geometries and their subspaces

Type 3: $U^\perp$ external: Then $\mathcal{Q}'$ is an elliptic quadric and $\Sigma(\mathcal{Q}')$ is a Möbius plane.

The associated algebra L is a quadratic field extension over K. It can be found as a subalgebra in $R = \mathrm{M}(2 \times 2, K)$. E.g., if $L = K(t)$ with $t^2 = s \in K$ then let

$$a + bt \longmapsto \begin{pmatrix} a & b \\ sb & a \end{pmatrix}$$

The corresponding line model in $\mathrm{PG}(3, K)$ is an elliptic linear congruence.

Chain geometries and their subspaces

Examples: Subspaces of $\Sigma(\mathcal{Q})$, where $\mathcal{Q}$ is the Klein quadric:

Let U be 4-dimensional projective subspace (a hyperplane) of $\mathrm{PG}(5, K)$. Then $\mathcal{Q}' = \mathcal{Q} \cap U$ gives rise to a subspace of $\Sigma(\mathcal{Q})$.

There are two types: U is tangent or not.

Chain geometries and their subspaces

Type 1: U tangent hyperplane: Then $\mathcal{Q}'$ is a cone over some quadric in a 3-space.

The associated subalgebra of $\mathrm{M}(2\times 2, K)$ is the algebra $\mathbb{T}$ of upper triangular matrices (also called algebra of ternions).

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Instead, one may also use the algebra of lower triangular matrices, which is conjugate to $\mathbb{T}$ via

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ b & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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The corresponding line model in $\mathrm{PG}(3, K)$ is a special linear complex, i.e. the set of all lines meeting a fixed given line.

Chain geometries and their subspaces

Remark: If S and T are isomorphic subalgebras of $R = \mathrm{M}(2 \times 2, K)$, then (by the Skolem-Noether theorem) they are conjugate in R.

This means that not only the associated subspaces of $\Sigma(K, R)$ are isomorphic, but the line models in $\mathrm{PG}(3, K)$ are projectively equivalent.

Chain geometries and their subspaces

Type 2: U non-tangent hyperplane: Then $\mathcal{Q}'$ is the Lie quadric.

There is no associated subalgebra of $\mathrm{M}(2 \times 2, K)$.

So $\Sigma(\mathcal{Q}')$ cannot be described as some $\Sigma(K, S)$.

The corresponding line model in $\mathrm{PG}(3, K)$ is a general linear complex, i.e. the set of all isotropic lines w.r.t. a symplectic polarity.

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Remark: The points and lines on the Lie quadric form an orthogonal generalized quadrangle, and the Klein correspondence gives an isomorphism onto the dual of a symplectic quadrangle.

Chain geometries and their subspaces

Subspaces defined by Jordan systems

Let J be a strong Jordan system in the K-algebra R. Then

$$\mathbb{P}(J) = \{R(1+ab, a) \mid a, b \in J\}$$

is a subspace of $\Sigma(K, R)$.

Chain geometries and their subspaces

Subspaces defined by Jordan systems

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Question: Is the condition “strong” needed?

If J is strong, then

$$\mathbb{P}(J) = \{R(1 + ab, a) \mid a \in J, b \in J^*\} = \{R(a, 1 + ab) \mid a, b \in J\}$$

Chain geometries and their subspaces

Conversely:

Theorem. (Herzer 1992). Let S be a strong subspace of $\Sigma(K,R)$. Then there are a strong Jordan system J in R and a matrix $M \in \mathrm{GL}(2,R)$ such that

$$S = \mathbb{P}(J)M = \{R(1+ab,a)M \mid a,b \in J\}.$$

Remark: If the algebra R is strong, then each subspace of $\Sigma(K,R)$ is strong.

Chain geometries and their subspaces

Example: Let J be the Jordan system of all symmetric matrices in $R = \mathrm{M}(2 \times 2, K)$, with $|K| \geq 5.$.

Then $\mathbb{P}(J)$ is isomorphic the subspace of $\Sigma(\mathcal{Q})$ given by the Lie quadric.

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Remark: J is strong if $|K| \geq 5$: For $A \in J$, $k \in K$ we have

$$k + A = kI + A \notin J^* \iff \det(kI + A) = 0,$$

and this quadratic equation in k has most two solutions.

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- Jordan systems and related algebraic structures
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- **Quadric chain spaces**
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Quadric chain spaces

Let $\Sigma(\mathcal{Q}) = (\mathbb{P}(\mathcal{Q}), \mathcal{C}(\mathcal{Q}))$ be a quadric chain space as above. The conditions on the quadric $\mathcal{Q}$ imply that it can be described as follows:

The underlying vector space is $V \times K \times K$, and

$$\mathcal{Q} = \{K(v,x,y) \mid (v,x,y) \neq (0,0,0), Q(v) = xy\},$$

where Q is a quadratic form on V for which there exists a $w \in V$ with $Q(w) = 1$.

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Example: For $V = K^4$, $Q(v_1, v_2, v_3, v_4) = v_1 v_2 - v_3 v_4$, we get the Klein quadric.

Quadric chain spaces

Theorem. (A.B. 1997). Let $|K| \geq 5$. Then the chain space $\Sigma(\mathcal{Q})$ is isomorphic to the subspace $\mathbb{P}(J)$ of the chain geometry $\Sigma(K, R)$, where R is the Clifford algebra $\mathrm{Cl}(V, Q)$ and J is the Jordan system $J = Vw$ in R.

Quadric chain spaces

The **Clifford algebra** $R = \mathrm{Cl}(V, Q)$ is obtained as follows: If $b_1, b_2, \ldots, b_n$ is a basis of V, then

$$1, b_1, b_2, \ldots, b_n, b_1b_2, b_1b_3, \ldots, b_1b_n, \ldots, \ldots, b_1b_2 \cdots b_n$$

is a basis of R (so R has dimension 2^n), and the multiplication is determined by the rules

$$\forall v, u \in V : v^2 = Q(v),\ uv + vu = Q(u+v) - Q(u) - Q(v).$$

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In particular, V is a subspace of the vector space R, but V is not closed w.r.t. multiplication, and $1 \notin V$. Moreover, $v \in V$ is invertible $\Leftrightarrow Q(v) \neq 0$, because then $v \cdot \frac{1}{Q(v)} v = \frac{1}{Q(v)} v^2 = 1$.

Quadric chain spaces

$J = Vw$ (with $Q(w) = 1$) is a Jordan system in $R = \mathrm{Cl}(V, Q)$:

- $1 = Q(w) = ww \in J$
- $a = vw \in J^* \Longrightarrow$

  $a^{-1} = w^{-1}v^{-1} = wQ(v)^{-1}v = Q(v)^{-1}wv =$

  $= Q(v)^{-1}(Q(v+w) - Q(w) - Q(v) - vw) =$

  $= Q(v)^{-1}\big((Q(v+w) - Q(w) - Q(v)) \cdot 1 - vw\big) \in J.$

Quadric chain spaces

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The Clifford algebra $\mathrm{Cl}(V,Q)$ is $R = K1 + V + \ldots$, where we take

$$V = \left\{ \begin{pmatrix} v_3 & v_1 \\ v_2 & -v_3 \end{pmatrix} \mid v_i \in K \right\}, \quad Q(v) = -\det v.$$

Then $v \cdot v = Q(v) \cdot 1$.

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Then $v \cdot v = Q(v) \cdot 1$. For $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in V$ we have $Q(w) = 1$ and

$$J = Vw = \left\{ \begin{pmatrix} v_1 & v_3 \\ -v_3 & v_2 \end{pmatrix} \mid v_i \in K \right\},$$

which is isotopic to the Jordan system of symmetric matrices.

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Thank you for your attention !