ZIF LECTURES
LECTURE II
Seven qubits, octonions and the Fano plane

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Abstract

- Here we extend the black hole/qubit correspondence to the maximally supersymmetric $N = 8$ supergravities in $D = 4$ and $D = 5$

Lecture II: Seven qubits, octonions and the Fano plane

- $N = 8$ generalization
- E_7 and seven qubits
- Octonions and the Fano plane
- Classification of $N = 8$ black holes and entanglement
- Five-dimensional black holes and qutrits
- Recent developments

$N = 8$ GENERALIZATION

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Supergravity in $D \leq 11$

D	scalars/vectors	G	H
10A	1 / 1	$SO(1,1,R)$	–
10B	2 / 0	$SL(2,R)$	$SO(2,R)$
9	3 / 3	$SL(2,R) \times SO(1,1,R)$	$SO(2,R)$
8	7 / 6	$SL(2,R) \times SL(3,R)$	$SO(2,R) \times SO(3,R)$
7	14 / 10	$SL(5,R)$	$SO(5,R)$
6	25 / 16	$SO(5,5,R)$	$SO(5,R) \times SO(5,R)$
5	42 / 27	$E_{6(6)}(R)$	$USP(8)$
4	70 / 28	$E_{7(7)}(R)$	$SU(8)$
3	128/ 0	$E_{8(8)}(R)$	$SO(16,R)$

Table: The symmetry groups (G) of the low energy supergravity theories with 32 supercharges in different dimensions (D) and their maximal compact subgroups (H).

Embeddings

The $N = 2$ STU solution can usefully be embedded in

- $N = 4$ supergravity with symmetry $SL(2) \times SO(6, 22)$, where the charges transform as a $(2, 28)$.
- $N = 8$ supergravity with symmetry $E_{7(7)}$, where the charges transform as a 56.

Remarkably, the same five parameters suffice to describe these 56-charge black holes.

$E_{7(7)}$ and seven qubits

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$E_{7(7)}$

- There is, in fact, a quantum information theoretic interpretation of the 56 charge $N = 8$ black hole in terms of a Hilbert space consisting of seven copies of the three-qubit Hilbert space. It relies on the decomposition $E_{7(7)} \supset [SL(2)]^7$

Decomposition of the 56

- Under
$$E_{7(7)} \supset$$
$$SL(2)_A \times SL(2)_B \times SL(2)_C \times SL(2)_D \times SL(2)_E \times SL(2)_F \times SL(2)_G$$
the 56 decomposes as
$$56 \rightarrow$$
$$\begin{aligned} &(2,2,1,2,1,1,1) \\ +&(1,2,2,1,2,1,1) \\ +&(1,1,2,2,1,2,1) \\ +&(1,1,1,2,2,1,2) \\ +&(2,1,1,1,2,2,1) \\ +&(1,2,1,1,1,2,2) \\ +&(2,1,2,1,1,1,2) \end{aligned}$$

Seven qubits

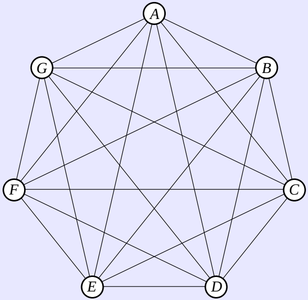
- It admits the interpretation of a tripartite entanglement of seven qubits, Alice, Bob, Charlie, Daisy, Emma, Fred and George:

$$
|\psi\rangle = \begin{array}{l} a_{ABD}|ABD\rangle \\ + b_{BCE}|BCE\rangle \\ + c_{CDF}|CDF\rangle \\ + d_{DEG}|DEG\rangle \\ + e_{EFA}|EFA\rangle \\ + f_{FGB}|FGB\rangle \\ + g_{GAC}|GAC\rangle \end{array}
$$

E_7 Entanglement

The following diagram may help illustrate the tripartite entanglement between the 7 qubits

E_7 Entanglement



Cartan invariant

- The entanglement measure given by Cartan's quartic $E_{7(7)}$ invariant.

$$I_4 = -\mathrm{Tr}((xy)^2) + \frac{1}{4}\mathrm{Tr}(xy)^2 - 4\,(\mathrm{Pf}(x) + \mathrm{Pf}(y))$$

x^{IJ} and y_{IJ} are again 8×8 antisymmetric charge matrices
Duff and Ferrara: quant-ph/0609227
Levay: hep-th/0610314

x^{IJ}

$$x^{IJ} =$$

$$\begin{pmatrix}
0 & -a_{111} & -b_{111} & -c_{111} & -d_{111} & -e_{111} & -f_{111} & -g_{111} \\
a_{111} & 0 & f_{001} & d_{100} & -c_{010} & g_{010} & -b_{100} & -e_{001} \\
b_{111} & -f_{001} & 0 & g_{001} & e_{100} & -d_{010} & a_{010} & -c_{100} \\
c_{111} & -d_{100} & -g_{001} & 0 & a_{001} & f_{100} & -e_{010} & b_{010} \\
d_{111} & c_{010} & -e_{100} & -a_{001} & 0 & b_{001} & g_{100} & -f_{010} \\
e_{111} & -g_{010} & d_{010} & -f_{100} & -b_{001} & 0 & c_{001} & a_{100} \\
f_{111} & b_{100} & -a_{010} & e_{010} & -g_{100} & -c_{001} & 0 & d_{001} \\
g_{111} & e_{001} & c_{100} & -b_{010} & f_{010} & -a_{100} & -d_{001} & 0
\end{pmatrix}$$

y_{IJ}

$$y_{IJ} = \begin{pmatrix} 0 & -a_{000} & -b_{000} & -c_{000} & -d_{000} & -e_{000} & -f_{000} & -g_{000} \\ a_{000} & 0 & f_{110} & d_{011} & -c_{101} & g_{101} & -b_{011} & -e_{110} \\ b_{000} & -f_{110} & 0 & g_{110} & e_{011} & -d_{101} & a_{101} & -c_{011} \\ c_{000} & -d_{011} & -g_{110} & 0 & a_{110} & f_{011} & -e_{101} & b_{101} \\ d_{000} & c_{101} & -e_{011} & -a_{110} & 0 & b_{110} & g_{011} & -f_{101} \\ e_{000} & -g_{101} & d_{101} & -f_{011} & -b_{110} & 0 & c_{110} & a_{011} \\ f_{000} & b_{011} & -a_{101} & e_{101} & -g_{011} & -c_{110} & 0 & d_{110} \\ g_{000} & e_{110} & c_{011} & -b_{101} & f_{101} & -a_{011} & -d_{110} & 0 \end{pmatrix}$$

I_4

Schematically,

$$
\begin{aligned}
I_4 &= a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4 \\
&+ 2\Big[a^2b^2 + a^2c^2 + a^2d^2 + a^2e^2 + a^2f^2 + a^2g^2 \\
&\quad + b^2c^2 + b^2d^2 + b^2e^2 + b^2f^2 + b^2g^2 \\
&\quad + c^2d^2 + c^2e^2 + c^2f^2 + c^2g^2 \\
&\quad + d^2e^2 + d^2f^2 + d^2g^2 \\
&\quad + e^2f^2 + e^2g^2 \\
&\quad + f^2g^2\Big] \\
&+ 8\left[abce + bcdf + cdeg + defa + efgb + fgac + gabd\right],
\end{aligned}
$$

where a^4 is Cayley's hyperdeterminant etc

$N = 8$ case

- Remarkably, because the generating solution depends on the same five parameters as the *STU* model, its classification of states will exactly parallel that of the usual three qubits. Indeed, the Cartan invariant reduces to Cayley's hyperdeterminant in a canonical basis.

Kallosh and Linde: hep-th/0602061

OCTONIONS AND THE FANO PLANE

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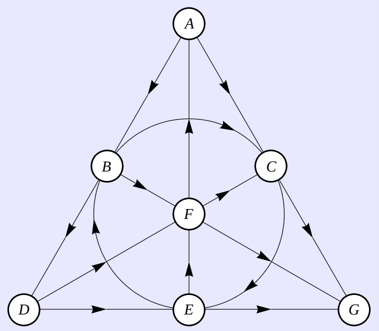
Fano plane

An alternative description is provided by the Fano plane which has seven points, representing the seven qubits, and seven lines (the circle counts as a line) with three points on every line, representing the tripartite entanglement, and three lines through every point.

Fano plane



Octonions

The Fano plane also provides the multiplication for the imaginary octonions:

	A	B	C	D	E	F	G
A		D	G	$-B$	F	$-E$	$-C$
B	$-D$		E	A	$-C$	G	$-F$
C	$-G$	$-E$		F	B	$-D$	A
D	B	$-A$	$-F$		G	C	$-E$
E	$-F$	C	$-B$	$-G$		A	D
F	E	$-G$	D	$-C$	$-A$		B
G	C	F	$-A$	E	$-D$	$-B$	

CLASSIFICATION

CLASSIFICATION

CLASSIFICATION

- Furthermore, one can relate the classification of three-qubit entanglements to the classification of supersymmetric black holes as in the following table:

Table

Class	S_A	S_B	S_C	Det a	Black hole	Susy
A-B-C	0	0	0	0	small	1/2
A-BC	0	> 0	> 0	0	small	1/4
B-CA	> 0	0	> 0	0	small	1/4
C-AB	> 0	> 0	0	0	small	1/4
W	> 0	> 0	> 0	0	small	1/8
GHZ	> 0	> 0	> 0	< 0	large	1/8
GHZ	> 0	> 0	> 0	> 0	large	0

Table: Classification of three-qubit entanglements and their corresponding $D = 4$ black holes.

Wrapped D3-branes and 3 qubits

WRAPPED M2-BRANES AND 2 QUTRITS

Qutrit interpretation

- All this suggests that the analogy between $D = 5$ black holes and three-state systems (0 or 1 or 2), known as qutrits [Duff and Ferrara: 0704.0507 [hep-th]], should involve the choice of wrapping a brane around one of three circles in T^3. This is indeed the case, with the number of qutrits being two.
- The two-qutrit system (where $A, B = 0, 1, 2$) is described by the state

$$|\Psi\rangle = a_{AB}|AB\rangle,$$

and the Hilbert space has dimension $3^2 = 9$.

2-tangle

- The bipartite entanglement of Alice and Bob is given by the 2-tangle
$$\tau_{AB} = 27\det \rho_A = 27|\det a_{AB}|^2,$$
where ρ_A is the reduced density matrix
$$\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi|.$$
- The determinant is invariant under $SL(3)_A \times SL(3)_B$, with a_{AB} transforming as a $(3, 3)$, and under a discrete duality that interchanges A and B.

$D = 5$ black hole

- For subsequent comparison with the $D = 5$ black hole, we restrict our attention to unnormalized states with real coefficients a_{AB}.
- There are three algebraically independent invariants : τ_{AB}, C_2 (the sum of the principal minors of ρ_{AB}) and the norm $\langle\Psi|\Psi\rangle$, corresponding to the dimension of

$$\frac{\mathbb{R}^3 \times \mathbb{R}^3}{SO(3) \times SO(3)}$$

namely, $9 - 6 = 3$.

$D = 5$ black hole

- Hence, the most general two-qutrit state can be described by just three parameters, which may conveniently taken to be three real numbers $N_0, N_1, N_2,$.

$$|\Psi\rangle = N_0|00\rangle + N_1|11\rangle + N_2|22\rangle$$

- A classification of two-qutrit entanglements, depending on the rank of the density matrix, is given in the following table:

$D = 5$ table

Class	C_2	τ_{AB}	Black hole	Susy
A-B	0	0	small	1/2
Rank 2 Bell	> 0	0	small	1/4
Rank 3 Bell	> 0	> 0	large	1/8

Table: Classification of two-qutrit entanglements and their corresponding $D = 5$ black holes.

$D = 5$ black hole

- The 9-charge $N = 2$, $D = 5$ black hole may also be embedded in the $N = 8$ theory in different ways. The most convenient microscopic description is that of three M2-branes wrapping the (58), (69), (710) cycles of the T^6 compactification of $D = 11$ M-theory, with wrapping numbers N_0, N_1, N_2 and intersecting over a point [Papadopoulos and Townsend: hep-th/9603087; Klebanov and Tseytlin:hep-th/9604166].
- To make the black hole/qutrit correspondence we associate the two T^3 with the $SL(3)_A \times SL(3)_B$ of the two qutrits Alice and Bob, where $|0\rangle$ corresponds to xoo, $|1\rangle$ to oxo and $|2\rangle$ to oox.The 9 different cycles then yield the 9 different basis vectors $|AB\rangle$ as in the last column of the following Table:

$D = 5$ table

5	6	7	8	9	10	macro charges	micro charges	$\|AB\rangle$
x	o	o	x	o	o	p^0	N_0	$\|00\rangle$
o	x	o	o	x	o	p^1	N_1	$\|11\rangle$
o	o	x	o	o	x	p^2	N_2	$\|22\rangle$
x	o	o	o	x	o	p^3	0	$\|01\rangle$
o	x	o	o	o	x	p^4	0	$\|12\rangle$
o	o	x	x	o	o	p^5	0	$\|20\rangle$
x	o	o	o	o	x	p^6	0	$\|02\rangle$
o	x	o	x	o	o	p^7	0	$\|10\rangle$
o	o	x	o	x	o	p^8	0	$\|21\rangle$

- We see immediately that we reproduce the three parameter two-qutrit state $|\Psi\rangle$:

$$|\Psi\rangle = N_0|00\rangle + N_1|11\rangle + N_2|22\rangle$$

- The black hole entropy, both macroscopic and microscopic, turns out to be given by the 2-tangle

$$S = 2\pi\sqrt{|\det a_{AB}|},$$

  and the classification of the two-qutrit entanglements matches that of the black holes .

- Note that the non-vanishing cubic combinations appearing in $\det a_{AB}$ correspond to groups of 3 wrapping cycles with no crosses in common, i.e. that intersect over a point.

Embeddings

- There is, in fact, a quantum information theoretic interpretation of the 27 charge $N = 8, D = 5$ black hole in terms of a Hilbert space consisting of three copies of the two-qutrit Hilbert space. It relies on the decomposition $E_{6(6)} \supset [SL(3)]^3$ and admits the interpretation of a bipartite entanglement of three qutrits, with the entanglement measure given by Cartan's cubic $E_{6(6)}$ invariant.
  Duff and Ferrara: 0704.0507 [hep-th]
- Once again, however, because the generating solution depends on the same three parameters as the 9-charge model, its classification of states will exactly parallel that of the usual two qutrits. Indeed, the Cartan invariant reduces to $\det a_{AB}$ in a canonical basis.
  Ferrara and Maldacena: hep-th/9706097

Summary

- Our M-theory analysis of the $D = 5$ black hole has provided an explanation for the appearance of the qutrit three-valuedness (0 or 1 or 2) that was lacking in the previous treatments: The brane can wrap one of the three circles in each T^3.
- The number of qutrits is two because of the number of extra dimensions is six.
- The three parameters of the real two-qutrit state are seen to correspond to three intersecting M2-branes.

RECENT DEVELOPMENTS

Recent developments

- Whether or not there is an underlying physical connection, this two-way process teaches us new things about both black holes and QI.
- Recent examples are:

FTS

- Correspondence between a three-qubit state vector ψ and a Freudenthal triple system Ψ over the Jordan algebra $C \oplus C \oplus C$:

$$\psi = a_{ABC}|ABC> \quad \leftrightarrow$$

$$\Psi = \begin{pmatrix} a_{111} & (a_{001}, a_{010}, a_{100}) \\ (a_{110}, a_{101}, a_{011}) & a_{000} \end{pmatrix}, \qquad (1)$$

the structure of the FTS naturally captures the SLOCC classification.
L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim and W. Rubens, arXiv:0812.3322 [quant-ph].

Class	Rank	FTS rank condition	
		vanishing	non-vanishing
Null	0	Ψ	–
A-*B*-*C*	1	$3T(\Psi, \Psi, \Phi) + \{\Psi, \Phi\}\Psi$	Ψ
A-*BC*	2a	$T(\Psi, \Psi, \Psi)$	γ^A
B-*CA*	2b	$T(\Psi, \Psi, \Psi)$	γ^B
C-*AB*	2c	$T(\Psi, \Psi, \Psi)$	γ^C
W	3	$q(\Psi)$	$T(\Psi, \Psi, \Psi)$
GHZ	4	–	$q(\Psi)$

Table: The entanglement classification of three qubits as according to the FTS rank system

Orbits

Table: Coset spaces of the orbits of the 3-qubit state space $C^2 \times C^2 \times C^2$ under the action of the SLOCC group $[SL(2, C)]^3$.

Class	FTS Rank	Orbits	dim
Separable	1	$\dfrac{[SL(2,C)]^3}{[SO(2,C)]^2 \ltimes C^3}$	4
Bi-separable	2	$\dfrac{[SL(2,C)]^3}{O(3,C) \times C}$	5
W	3	$\dfrac{[SL(2,C)]^3}{C^2}$	7
GHZ	4	$\dfrac{[SL(2,C)]^3}{[SO(2,C)]^2}$	7

New duality for black holes; arXiv:0903.5517

- It is well-known that the quantized charges x of 4D black holes may be assigned to elements of an integral Freudenthal triple system (FTS) whose automorphism group is the corresponding U-duality. The FTS is equipped with a quartic form $\Delta(x)$ whose square root yields the lowest order black hole entropy.
- We show that a subset of these black holes, for which $\Delta(x)$ is necessarily a perfect square, admit a *Freudenthal dual* with integer charges $\tilde{x}$, for which $\tilde{\tilde{x}} = -x$ and $\Delta(\tilde{x}) = \Delta(x)$ Some, but not all, of other discrete U-duality invariants are also Freudenthal invariant.
- Similar story in 5D where we introduce a *Jordan dual* $A^\star$, for which $A^{\star\star} = A$ with cubic norm $N(A^\star) = N(A)$, whose square is necessarily a perfect cube.

Octonions and supersmmetry

- In the $\mathcal{N} = 8$ case, the **56** of $E_{7(7)}$ decomposes as

$$\mathbf{56} \to (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32}), \tag{2}$$

  under

$$E_{7(7)} \supset SL(2) \times SO(6,6) \tag{3}$$

  where $SL(2)$ is the electric-magnetic S-duality and $SO(6,6)$ is the T-duality group.

- The $(\mathbf{2}, \mathbf{12})$ is identified as the NS-NS sector where as the $(\mathbf{1}, \mathbf{32})$ is associated with the R-R charges.
- In the Fano plane picture going from NS to NS+RR is going from quaternions to octonions. Suggestive of hidden role of octonions in M-theory?

Superqubits

- We provide a supersymmetric generalisation of n quantum bits by extending the LOCC entanglement equivalence group $[SU(2)]^n$ to the supergroup $[uOSp(2|1)]^n$ and the SLOCC equivalence group $[SL(2, C)]^n$ to the supergroup $[OSp(2|1)]^n$.
- We introduce the appropriate supersymmetric generalisations of the conventional entanglement measures for the cases of $n = 2$ and $n = 3$.
- In particular, super-GHZ states are characterised by a non-vanishing superhyperdeterminant.
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