Dual Polar Spaces and the Geometry of Matrices

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TU WIEN
Technische Universität Wien
Vienna University of Technology
Differentialgeometrie und Geometrische Strukturen

Hans Havlicek
Forschungsgruppe Differentialgeometrie und Geometrische Strukturen
Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien
havlicek@geometrie.tuwien.ac.at

Part 1

Rectangular Matrices

The first part deals with some basic notions and results from the Geometry of Rectangular Matrices. Square matrices are not excluded, and their particular properties will be exhibited in due course.

Our exposition follows the book of Z.-X. Wan [22].

Basic Notions

- Let F be a field (not necessarily commutative) or, said differently, a division ring.
- We denote by F^n the left vector space of row vectors $x = (x_1, x_2, \ldots, x_n)$ with entries from F.
- Let $F^{m\times n}$, $m, n \geq 1$, be the set of all $m \times n$ matrices over a division ring F.

There is yet no structure on the set $F^{m\times n}$.

A Single Matrix

- Each matrix $A \in F^{m \times n}$ determines a linear mapping

$$f_A : F^m \to F^n : x \mapsto xA.$$

- All linear mappings $F^m \to F^n$ arise in this way.

- The left row space of A is the subspace of F^n which is generated by the rows of A. It equals the image of the linear mapping f_A.

- The dimension of the left row space of A is called the left row rank of A.

The Dual Approach

Each column vector (single column matrix) $a^* \in F^{m\times 1} =: F^{m*}$ determines a linear form $F^m \to F : x \mapsto x \cdot a^*$. The elements of F^{m*} can be identified with the dual vector space of F^m, which is a right vector space over F.

This yields our second interpretation: Any matrix $A \in F^{m\times n}$ determines a linear mapping between dual vector spaces, viz.

$$f_A^{\mathrm{T}} : F^{n*} \to F^{m*} : y^* \mapsto Ay^*$$

which is known as the transpose (or dual) of the mapping $f_A : x \mapsto xA$.
We obtain, mutatis mutandis, the notions right column space and right column rank of A.

Remarks

For any matrix one may introduce four notions of rank (left / right, row / column).

- The left row rank equals the right column rank of A. Either of these numbers will simply be called the rank of A, in symbols $\operatorname{rk} A$.

- The right row rank equals the left column rank of A.
  We shall not make use of these ranks.

- The left row rank and the right row rank of A may be different.

Example The matrix

$$\begin{pmatrix} 1 & j \\ i & k \end{pmatrix}$$

over the real quaternions $\mathbb{H}$ has left row rank 1 and right row rank 2, because

$$i(1, j) = (i, k), \quad \text{whereas} \quad (1, j)i = (i, -k) \neq (i, k).$$

Vector Space on $F^{m\times n}$

The sum of two matrices $A, B \in F^{m\times n}$ corresponds in a natural way to the sum of the associated mappings $f_A + f_B$ (and dually).

Even though a matrix A can be multiplied by a scalar $\lambda \in F$ from the left hand side (λA) or the right hand $(A\lambda)$, these products are in general not useful in terms of our interpretations of matrices as linear mappings:

"The λ is never where it should be!"

Only when λ is in the centre of F, in symbols $\lambda \in Z(F)$, then $\lambda A = A\lambda$ may be viewed as the product of λ and either of the two linear mappings given by A:

$$(\lambda f_A) : x \mapsto \lambda(xA) = x(\lambda A), \qquad (f_A^{\mathrm{T}}\lambda) : y^* \mapsto (Ay^*)\lambda = (\lambda A)y^*.$$

Hence $F^{m\times n}$ is a (left or right) vector space over $Z(F)$. This will be of some importance in what follows.

Rank One Matrices

Given a column vector $a^* = (a_1^*, a_2^*, \ldots, a_m^*)^{\mathrm{T}}$ (i. e. a linear form on F^m) and a vector $c = (c_1, c_2, \ldots, c_n)$ we obtain the linear mapping

$$F^m \longrightarrow F^n : x \mapsto x \cdot a^* \cdot c.$$

Its matrix is therefore

$$a^* \cdot c = \begin{pmatrix} a_1^* c_1 & a_1^* c_2 & \ldots & a_1^* c_m \\ a_2^* c_1 & a_2^* c_2 & \ldots & a_2^* c_m \\ \multicolumn{4}{c}{\dotfill} \\ a_n^* c_1 & a_n^* c_2 & \ldots & a_n^* c_m \end{pmatrix}.$$

This matrix has rank one provided that $a^* \neq 0$ and $c \neq 0$. All matrices with rank ≤ 1 arise in this way.

Graph on $F^{m\times n}$

Let $F^{m\times n}$, $m, n \geq 2$, be the set of all $m \times n$ matrices over a field F. Hence $F^{m\times n}$ contains matrices of rank ≥ 2.

- Two matrices A and B are called *adjacent* if $A - B$ is of rank one.
- We consider $F^{m\times n}$ as the set of vertices of an undirected graph the edges of which are precisely the (unordered) pairs of adjacent matrices.
- Two matrices A and B are at the graph-theoretical distance $k \geq 0$ if, and only if,

$$\mathrm{rk}(A - B) = k.$$

Almost a “Middle Product”

Given $a^* \in F^{m*} \setminus \{0\}$, $c \in F^n \setminus \{0\}$, and $\lambda \in F$ one may “multiply the rank one matrix $A := a^*c$ by $\lambda \in F$ from the middle” as follows:

$$(a^*\lambda)c = a^*(\lambda c) =: a^*\lambda c$$

This “product” in general depends on the vectors which are chosen to factorise A. Indeed, we have

$$A = (a^*\alpha)(\alpha^{-1}c) \quad \text{for all} \quad \alpha \in F \setminus \{0\},$$

and

$$(a^*\alpha)\lambda(\alpha^{-1}c) = a^*(\alpha\lambda\alpha^{-1})c.$$

Nevertheless, the set of matrices

$$\{a^*\lambda c \mid \lambda \in F\}$$

depends only on the rank one matrix A and the ground field F.

Lines

Given $a^* \in F^{m*} \setminus \{0\}$, $c \in F^n \setminus \{0\}$ and any matrix $R \in F^{m\times n}$ the set

$$\{a^*\lambda c + R \mid \lambda \in F\}$$

is called a *LINE* of $F^{m\times n}$.

Let $\mathcal{L}$ be the set of all such lines. Then $\left(F^{m\times n}, \mathcal{L}\right)$ is a partial linear space, called the *space of $m \times n$ matrices over F.*
In this context the elements of $F^{m\times n}$ will also be called *POINTS*.

Two matrices A and B are adjacent if, and only if, they are distinct and COLLINEAR.
In this case the unique LINE joining A and B equals $\{A, B\}^{\sim\sim}$, where

$$\mathcal{M}^{\sim} := \{X \mid \forall\, Y \in \mathcal{M} : X \text{ is adjacent or equal to } Y\}.$$

Example

We consider the real quaternions $\mathbb{H}$. The LINE joining the 2×2 zero matrix and the matrix

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 & i \end{pmatrix} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} =: A$$

equals the set of all matrices

$$\begin{pmatrix} 1 \cdot \lambda \cdot 1 & 1 \cdot \lambda \cdot i \\ i \cdot \lambda \cdot 1 & i \cdot \lambda \cdot i \end{pmatrix} = \begin{pmatrix} \lambda & \lambda i \\ i\lambda & i\lambda i \end{pmatrix},$$

where λ ranges in $\mathbb{H}$. The matrices (POINTS) of this LINE are in general neither left proportional nor right proportional to A.

Example

We consider the space of 2×2 matrices over the Galois field $\mathrm{GF}(2)$. All its rank one matrices can be read off from the following table:

	$\begin{pmatrix} 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Thus there are nine LINES through the zero matrix, each comprising two POINTS. The space of 2×2 over $\mathrm{GF}(2)$ matrices is a *partial affine space*, viz. the affine space on $\mathrm{GF}(2)^{2\times 2}$ with six parallel classes of lines removed.

Summary

- The space $(F^{m\times n}, \mathcal{L})$ is a connected partial linear space.
- If F is a proper skew field then $F^{m\times n}$ can be considered as a vector space (affine space) over F from the left and right hand side, and (more naturally) as a vector space over the centre $Z(F)$. The LINES of $\mathcal{L}$ are in general not lines of any of these affine spaces.
- If F is a commutative field then $F^{m\times n}$ can be considered as a (left or right) vector space (affine space) over $F = Z(F)$. The LINES of $\mathcal{L}$ comprise some of the parallel classes of lines of this affine space.

Automorphisms

An *automorphism* of the space $(F^{m\times n}, \mathcal{L})$ is a bijection

$$\varphi : F^{m\times n} \to F^{m\times n} : X \mapsto X^{\varphi}$$

preserving adjacency in both directions. Consequently, LINES are mapped onto LINES under φ and φ^{-1}.

Examples

- Translations: $X \mapsto X + R$, where $R \in F^{m\times n}$.
- Equivalence transformations: $X \mapsto PXQ$, where $P \in \mathrm{GL}_m(F)$ and $Q \in \mathrm{GL}_n(F)$.
- Field automorphisms: $X \mapsto X^{\sigma}$, where σ is an automorphism of F acting on the entries of X.
- σ-Transpositions: $X \mapsto (X^{\sigma})^{\mathrm{T}}$, where σ is an antiautomorphism of F acting on the entries of X. (Only for $n = m$ provided that such a σ exists.)

Remarks on Automorphisms

- If $m = n$ and F is a commutative field then the transposition $X \mapsto X^{\mathrm{T}}$ is an automorphism.
- If $m = n$ and F is a proper skew field then $X \mapsto X^{\mathrm{T}}$ need not be automorphism. E. g., over the real quaternions $\mathbb{H}$ we already saw that

$$\operatorname{rk}\begin{pmatrix} 1 & j \\ i & k \end{pmatrix} = 1, \quad \text{whereas} \quad \operatorname{rk}\begin{pmatrix} 1 & j \\ i & k \end{pmatrix}^{\mathrm{T}} = \operatorname{rk}\begin{pmatrix} 1 & i \\ j & k \end{pmatrix} = 2.$$

- If $m = n$, F is a proper skew field, and σ is an antiautomorphism then $X \mapsto X^{\sigma}$ need not be an automorphism. E. g., letting $\sigma = \bar{\ }$ to be the conjugation of $\mathbb{H}$ gives

$$\operatorname{rk}\begin{pmatrix} 1 & j \\ i & k \end{pmatrix} = 1, \quad \text{whereas} \quad \operatorname{rk}\overline{\begin{pmatrix} 1 & j \\ i & k \end{pmatrix}} = \operatorname{rk}\begin{pmatrix} 1 & -j \\ -i & -k \end{pmatrix} = 2.$$

- There are proper skew fields without any antiautomorphism [4].

Fundamental Theorem

Theorem (L. K. Hua 1951 et al.) *Every bijective mapping*

$$\varphi : F^{m\times n} \to F^{m\times n} : X \mapsto X^{\varphi}$$

preserving adjacency in both directions is of the form

$$X \mapsto PX^{\sigma}Q + R,$$

where $P \in \mathrm{GL}_m(F)$, $Q \in \mathrm{GL}_n(F)$, $R \in F^{m\times n}$, and σ is an automorphism of F.

If $m = n$, then we have the additional possibility that

$$X \mapsto P(X^{\sigma})^{\mathrm{T}}Q + R$$

where P, Q, R are as above, σ is an antiautomorphism of F, and T denotes transposition.

The assumptions in Hua's fundamental theorem can be weakened.

W.-l. Huang and Z.-X. Wan [18], P. Šemrl [20].

Avoiding Matrices

From a theoretical viewpoint one may define the space of $m \times n$ matrices over F in a coordinate free way.

with coordinates / matrices	without coordinates / matrices
F^m	$V \ldots m$-dimensional left vector space over F
F^n	$W \ldots n$-dimensional left vector space over F
$F^{m\times n}$	$\mathrm{Hom}_F(V, W) \cong V^* \otimes_F W \ldots$ tensor product
$a^* \cdot c$	$a^* \otimes c \ldots$ pure tensor
rank of a matrix	rank of a linear mapping

Part 2

Grassmannians

We establish an embedding of any space of rectangular matrices in an appropriate Grassmann space. For square matrices this embedding will reveal neat connections with the projective lines over matrix rings.

Projective Space on F^{s+1}

Let $\mathrm{PG}(s, F)$ be the projective space over the left vector space F^{s+1}, where F is a field.

- In what follows we do not distinguish between subspaces of F^{s+1} and subspaces of $\mathrm{PG}(s, F)$.
- The *dimension* $\dim W$ of a subspace W is always understood as the "projective dimension", which is one less than the vector space dimension.
- Subspaces of dimension 0, 1, 2, 3, and $s-1$ are called *points*, *lines*, *planes*, *solids*, and *hyperplanes*, respectively.
- We use the shorthand *d-subspace* for a d-dimensional subspace.

Grassmann Graph on $\mathcal{G}_{s,d}$

Let $\mathcal{G}_{s,d}(F)$ be the Grassmannian of all d-subspaces of $\mathrm{PG}(s,F)$. We assume $1 \leq d \leq s-2$ in order to avoid trivial cases.

- Two d-subspaces W_1 and W_2 are called *adjacent* if $\dim W_1 \cap W_2 = d-1$.

- We consider $\mathcal{G}_{s,d}(F)$ as the set of vertices of an undirected graph the edges of which are the (unordered) pairs of adjacent d-subspaces.

- Two d-subspaces W_1 and W_2 are at graph theoretical distance $k \geq 0$ if, and only if,
$$\dim W_1 \cap W_2 = d-k.$$

- For any subset $\mathcal{M} \subset \mathcal{G}_{s,d}(F)$ we define
$$\mathcal{M}^{\sim} := \{X \mid \forall\, Y \in \mathcal{M} : X \text{ is adjacent or equal to } Y\}.$$

Grassmann Space on $\mathcal{G}_{s,d}$

Given a $(d-1)$-subspace U and a $(d+1)$-subspace V of $\mathrm{PG}(s,F)$ with $U \subset V$ the set

$$\{W \in \mathcal{G}_{s,d}(F) \mid U \subset W \subset V\}$$

is called a *pencil*.

The set $\mathcal{G}_{s,d}(F)$, considered as a set of *POINTS*, together with the set $\mathcal{P}$ of all its pencils, considered as its set of *LINES*, is called the *Grassmann space* of d-subspaces of $\mathrm{PG}(s,F)$.

The Grassmann space $(\mathcal{G}_{s,d}(F), \mathcal{P})$ is a connected partial linear space.

Two d-subspaces W_1 and W_2 are adjacent if, and only if, they are distinct and COLLINEAR. In this case the unique LINE joining W_1 and W_2 equals $\{W_1, W_2\}^{\sim\sim}$.

Fundamental Theorem

(W. L. Chow 1949) *Every bijective mapping*

$$\varphi : \mathcal{G}_{s,d}(F) \to \mathcal{G}_{s,d}(F) : X \mapsto X^{\varphi}$$

preserving adjacency in both directions is of the form

$$X \mapsto \{x^{\sigma} P \mid x \in X \subset F^{s+1}\},$$

where $P \in \mathrm{GL}_m(F)$ *and* σ *is an automorphism of* F.

If $s = 2d+1$, *then we have the additional possibility that*

$$X \mapsto \{y \in F^{s+1} \mid yP(x^{\sigma})^{\mathrm{T}} = 0 \text{ for all } x \in X \subset F^{s+1}\},$$

where P *is as above,* σ *is an antiautomorphism of* F, *and* T *denotes transposition.*

The assumptions in Chow's fundamental theorem can be weakened.

W.-l. Huang [11].

An Embedding

We adopt the assumptions from Part 1. The $m \times m$ identity matrix will be denoted by I_m. Horizontal augmentation of (suitable) matrices A, B is written as $A|B$.

$F^{m\times n}$ can be embedded in the Grassmannian $\mathcal{G}_{m+n-1,m-1}(F)$ as follows:

$$\begin{array}{ccccc} F^{m\times n} & \longrightarrow & F^{m\times(m+n)} & \longrightarrow & \mathcal{G}_{m+n-1,m-1}(F) \\ X & \longmapsto & X|I_m & \longmapsto & \text{left rowspace of } X|I_m \end{array}$$

- Matrices $X, Y \in F^{m\times n}$ are adjacent if, and only if, their images in $\mathcal{G}_{m+n-1,m-1}(F)$ are adjacent.
- LINES of matrices are mapped to LINES (pencils) of the Grassmann space with one element removed.

Projective Matrix Spaces

Each element of the Grassmannian $\mathcal{G}_{m+n-1,m-1}(F)$ can be viewed as the left row space of a matrix $X|Y$ with rank m, where $X \in F^{m\times n}$ and $Y \in F^{m\times m}$.

- $X|Y$ and $X'|Y'$ have the same left row space, if and only if, there is a $T \in \mathrm{GL}_m(F)$ with $X' = TX$ and $Y' = TY$.

- One may consider a pair $(X, Y) \in F^{m\times n} \times F^{m\times m}$ as left homogeneous coordinates of an element of $\mathcal{G}_{m+n-1,m-1}(F)$ provided that $\mathrm{rk}(X|Y) = m$.

  This means that $X|Y$ possesses an invertible $m \times m$ submatrix. (This submatrix need not be Y).

The Grassmann space on $\mathcal{G}_{m+n-1,m-1}(F)$ is often called the *projective space* of $m \times n$ matrices over F, even though it is not a projective space in the usual sense.

Points at Infinity

- A subspace with coordinates (X, Y) is in the image of the embedding

$$F^{m\times n} \to \mathcal{G}_{m+n-1,m-1}(F)$$

  if, and only if, Y is invertible. In this case its only preimage is the matrix $Y^{-1}X \in F^{m\times n}$.
- All subspaces with coordinates (X, Y), where $Y \notin \mathrm{GL}_m(F)$, are called *POINTS at infinity* of the Grassmann space.
  Clearly, this notion depends on the chosen embedding.
- There is a distinguished $(n-1)$-subspace of $\mathrm{PG}(m+n-1, F)$ given by the left row space of the $n \times (m+n)$ matrix $I_n|0$.
- An element of $\mathcal{G}_{m+n-1,m-1}(F)$ is at infinity, precisely when it has at least one common point with this $(n-1)$-subspace.

See also R. Metz [19].

Example

The space of 2×2 matrices over $\mathrm{GF}(2)$ comprises 16 elements. It can be embedded in the Grassmann space of lines in $\mathrm{PG}(3,2)$. Note that $\#\mathcal{G}_{3,1}(\mathrm{GF}(2)) = 35$.

There is a unique distinguished line, viz. the row space of $I_2|0$. There are

$$3 \cdot 6 + 1 = 19$$

lines which have at least one common point with this line. These are the POINTS at infinity of the Grassmann space.

The $35 - 19 = 16$ lines which are skew to the line with coordinates $(I_2, 0)$ are in one-one correspondence with the 16 matrices of $\mathrm{GF}(2)^{2\times 2}$.

Example

The space of 2×3 matrices over $\mathrm{GF}(2)$ comprises 64 elements. It can be embedded in the Grassmannian of lines in $\mathrm{PG}(4,2)$. Note that $\#\mathcal{G}_{4,1}(\mathrm{GF}(2)) = 155$.

There is a unique distinguished plane, viz. the row space of $I_3|0$. There are

$$7 \cdot 12 + 7 = 91$$

lines which have at least one common point with this plane. They are the POINTS at infinity of the Grassmann space.

The $155 - 91 = 64$ lines which are skew to the plane with coordinates $(I_3, 0)$ are in one-one correspondence with the 64 matrices of $\mathrm{GF}(2)^{2\times 3}$.

Square Matrices

We consider square matrices ($m = n \geq 2$) and the full matrix algebra $R := (F^{n\times n}, +, \cdot)$ over $Z(F)$.

In terms of our left-homogeneous coordinates $(X, Y) \in R^2$ the POINT set of the Grassmannian $\mathcal{G}_{2n-1,n-1}(F)$ is the same as the POINT set of the projective line $\mathbb{P}(R)$ over the full matrix algebra R (up to irrelevant differences). Cf. the lecture of A. Blunck or [2].

There is one difference though:

- The basic notion in the Grassmann space is adjacency: $\dim W_1 \cap W_2 = n - 2$.
- The basic notion in ring geometry is being distant: $\dim W_1 \cap W_2 = -1$.

Each of these relations can be expressed in terms of the other. A. Blunck, H. H. [1], W.-l. Huang, H. H. [15].

Hence the two structural approaches are essentially the same.

Part 3

Symmetric Matrices

The third part deals with some basic notions and results from the Geometry of Symmetric Matrices over a commutative field. Some results will depend on the characteristic of the ground field being two or not.

Our exposition follows the book of Z.-X. Wan [22].

Basic Notions

- Let F be a commutative field.
- Let $\mathrm{S}_n(F) \subset F^{n\times n}$, $n \geq 1$, be the set of all symmetric $n \times n$ matrices over F.
- If $\mathrm{Char}\, F \neq 2$ then the $n \times n$ zero matrix is the only alternating matrix which is also symmetric.
- If $\mathrm{Char}\, F = 2$ then any alternating $n \times n$ matrix is also symmetric. A symmetric matrix is non-alternating if, and only if, at least one of its diagonal entries is $\neq 0$.

The set $\mathrm{S}_n(F)$ is a subset of the matrix space $F^{n\times n}$.

A Single Symmetric Matrix

- Each symmetric matrix $A \in \mathrm{S}_n(F)$ determines a linear mapping

$$f_A : F^n \longrightarrow F^{n*} : y \longmapsto Ay^{\mathrm{T}}.$$

  This provides the link with Part 1.

  Moreover, the matrix A defines a symmetric bilinear form

$$g_A : F^n \times F^n \longrightarrow F : (x, y) \longmapsto xAy^{\mathrm{T}}.$$

  We shall adopt this interpretation of the matrix A.

- All symmetric bilinear forms $F^n \times F^n \longrightarrow F$ arise in this way.

- Since F is commutative, we may unambiguously speak of the rank of A.

Symmetric Rank One Matrices

Given a column vector $a^* = (a_1^*, a_2^*, \ldots, a_m^*)^{\mathrm{T}} \in F^{n*}$ we obtain the symmetric bilinear form

$$F^n \times F^n \to F : (x, y) \mapsto (x \cdot a^*)(y \cdot a^*) = x \cdot (a^* \cdot (a^*)^{\mathrm{T}}) \cdot y^{\mathrm{T}}.$$

Its matrix is therefore

$$a^* \cdot (a^*)^{\mathrm{T}} = \begin{pmatrix} a_1^* a_1^* & a_1^* a_2^* & \ldots & a_1^* a_n^* \\ a_2^* a_1^* & a_2^* a_2^* & \ldots & a_2^* a_n^* \\ \ldots & \ldots & \ldots & \ldots \\ a_n^* a_1^* & a_n^* a_2^* & \ldots & a_n^* a_n^* \end{pmatrix}.$$

This matrix has rank one provided that $a^* \neq 0$. All symmetric matrices with rank ≤ 1 arise in this way.

Vector Space on $\mathrm{S}_n(F)$

The sum of two symmetric matrices $A, B \in F^{n\times n}$ corresponds in a natural way to the sum of the associated bilinear forms $g_A + g_B$.

Since F coincides with its centre $Z(F)$, for any $\lambda \in F$ the (obviously symmetric) matrix $\lambda A = A\lambda$ may be viewed as the product of the scalar λ and the symmetric bilinear form g_A:

$$(\lambda g_A) : (x, y) \mapsto \lambda(xAy^{\mathrm{T}}) = x(\lambda A)y^{\mathrm{T}}.$$

Hence $\mathrm{S}_n(F)$ is a (left or right) vector space over F.

Graph on $\mathrm{S}_n(F)$

We assume $n \geq 2$. Hence $\mathrm{S}_n(F)$ contains matrices of rank ≥ 2.

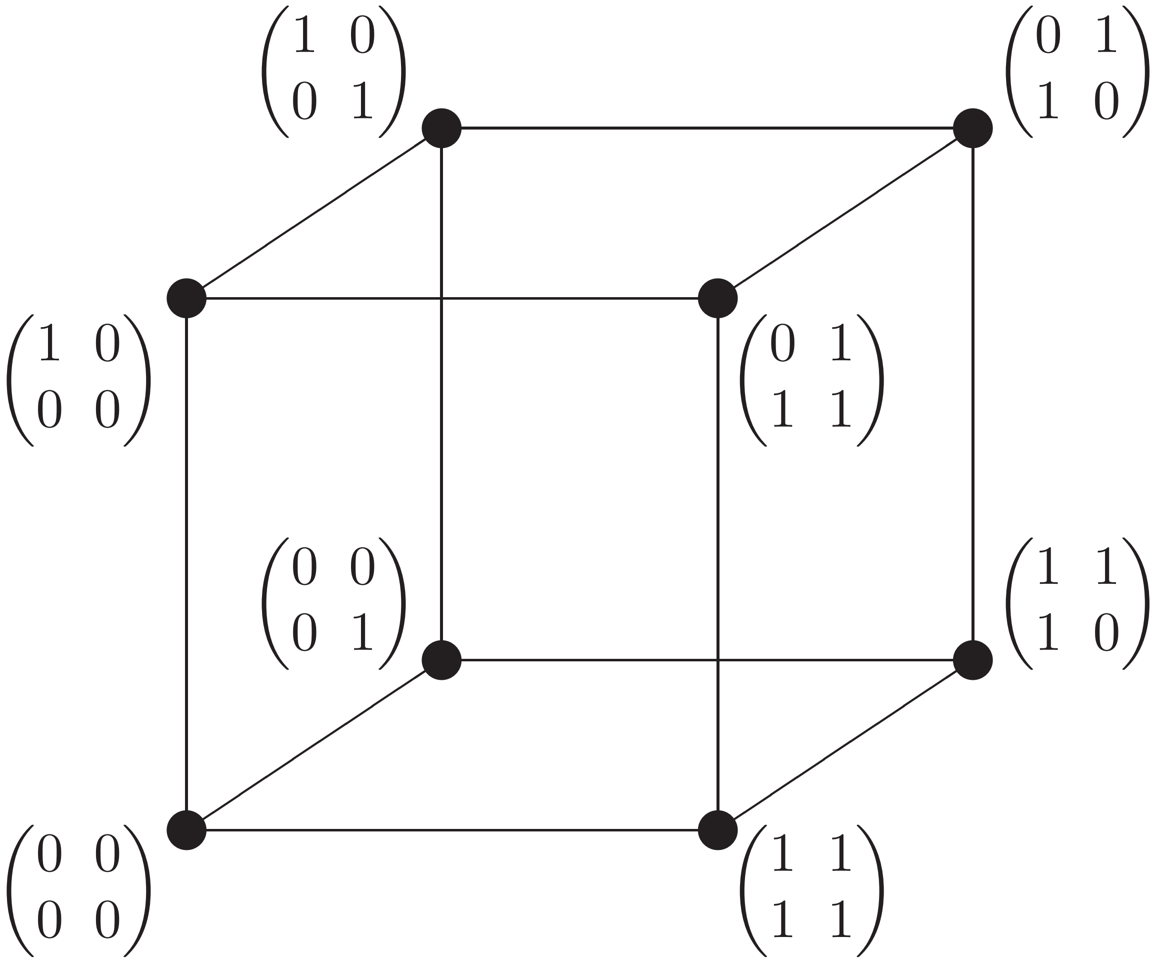
- The notion of adjacency is inherited form $F^{n\times n}$.
- We consider $\mathrm{S}_n(F)$ as the set of vertices of an undirected graph the edges of which are precisely the (unordered) pairs of adjacent symmetric matrices.
- Two symmetric matrices A and B are at the graph-theoretical distance $k \geq 0$ if, and only if,

$$k = \begin{cases} \mathrm{rk}(A-B) & \text{and} \quad A-B \quad \text{is non-alternating or zero,} \\ \mathrm{rk}(A-B)+1 & \text{and} \quad A-B \quad \text{is alternating and non-zero.} \end{cases}$$

  The second possibility occurs only for $\mathrm{Char}\, F = 2$ and $3 \leq k \leq n+1$, where k is odd.
- The diameter (maximal distance) in this graph is n or $n+1$. The second possibility occurs precisely when $\mathrm{Char}\, F = 2$ and n is even.

Example

The graph of symmetric 2×2 matrices over $\mathrm{GF}(2)$ can be illustrated as a cube:



The diameter of this graph is 3. Opposite points of the cube stand for points at distance 3.

Lines

Given $a^* \in F^{n*} \setminus \{0\}$ and any matrix $R \in \mathrm{S}_n(F)$ the set

$$\{\lambda a^*(a^*)^{\mathrm{T}} + R \mid \lambda \in F\}$$

is called a *LINE* of $\mathrm{S}_n(F)$.

Let $\mathcal{L}_{\mathrm{S}}$ be the set of all such LINES. Then $\big(\mathrm{S}_n(F), \mathcal{L}_{\mathrm{S}}\big)$ is a partial linear space, called the *space of symmetric $n \times n$ matrices over F.*

In this context the elements of $\mathrm{S}_n(F)$ will also be called *POINTS*.

Two symmetric matrices A and B are adjacent if, and only if, they are distinct and COLLINEAR. In this case the unique LINE joining A and B equals

$$\begin{aligned}\{A, B\}^{\sim} &= \{X \in \mathrm{S}_n(F) \mid (X = A) \text{ or } (X = B) \text{ or } (X \text{ is adjacent to } A \text{ and } B)\} \\ &= \{\lambda(A - B) + B \mid \lambda \in F\}.\end{aligned}$$

Example

We consider the space of symmetric 2×2 matrices over the Galois field $\mathrm{GF}(2)$. It contains the following three symmetric matrices with rank 1:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus there are three LINES through the zero matrix, each comprising two POINTS. The space of symmetric 2×2 matrices over $\mathrm{GF}(2)$ is a *partial affine space*, viz. the affine space on $\mathrm{S}_2(\mathrm{GF}(2))$ with $4 = 7 - 3$ parallel classes of lines removed.

Example

We consider the three-dimensional space-time $\mathbb{R}^3$ with the indefinite quadratic form given by the matrix $\mathrm{diag}(1,1,-1)$. The mapping

$$\gamma : \mathbb{R}^3 \to \mathrm{S}_2(\mathbb{R}) : (x_1, x_2, x_3) \mapsto \begin{pmatrix} x_2 + x_3 & x_1 \\ x_1 & -x_2 + x_3 \end{pmatrix}$$

is bijective and

$$-\det((x_1, x_2, x_3)^\gamma) = -\det \begin{pmatrix} x_2 + x_3 & x_1 \\ x_1 & -x_2 + x_3 \end{pmatrix} = x_1^2 + x_2^2 - x_3^2.$$

The light-like lines of $\mathbb{R}^3$ correspond under γ to the LINES of $\mathcal{L}_\mathrm{S}$ and vice versa.

Summary

- The space $(\mathrm{S}_n(F), \mathcal{L}_\mathrm{S})$ is a connected partial linear space.

- Since F is a commutative field, the set $\mathrm{S}_n(F)$ can be considered as a (left or right) vector space (affine space) over F. The LINES of $\mathcal{L}_\mathrm{S}$ comprise some of the parallel classes of lines of this affine space.

Remark: In the book of Wan [22] also another kind of subset of $\mathrm{S}_n(F)$ is called a "line". Subsets of this kind provide a powerful tool for proving the Fundamental Theorem of the Geometry of Symmetric Matrices in [22]. They will not be considered here.

Automorphisms

An *automorphism* of the space $(\mathrm{S}_n(F), \mathcal{L}_\mathrm{S})$ is a bijection

$$\varphi : \mathrm{S}_n(F) \to \mathrm{S}_n(F) : X \mapsto X^\varphi$$

preserving adjacency in both directions. Consequently, LINES are mapped onto LINES under φ and φ^{-1}.

Examples

- Translations: $X \mapsto X + R$, where $R \in \mathrm{S}_n(F)$.
- Congruence transformations: $X \mapsto PXP^\mathrm{T}$, where $P \in \mathrm{GL}_n(F)$.
- Field automorphisms: $X \mapsto X^\sigma$, where σ is an automorphism of F acting on the entries of X.
- Scalings: $X \mapsto \lambda X$, where $\lambda \in F \setminus \{0\}$.

All these automorphisms have the property

$$\mathrm{rk}(X - Y) = \mathrm{rk}(X^\varphi - Y^\varphi) \quad \text{for all} \quad X, Y \in \mathrm{S}_n(F).$$

An Exceptional Automorphism

The following mapping is an automorphism of $\mathrm{S}_3(\mathrm{GF}(2))$:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix},$$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 1 \\ x_{13} & 1 & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11}+1 & x_{12}+1 & x_{13}+1 \\ x_{12}+1 & x_{22} & 1 \\ x_{13}+1 & 1 & x_{33} \end{pmatrix}. \qquad (*)$$

The mapping $(*)$ is an involution fixing 32 out of the 64 matrices of $\mathrm{S}_2(\mathrm{GF}\,2))$. The zero matrix is fixed, but $(*)$ is not rank preserving, since some alternating matrices with rank two are mapped to non-alternating with rank three. For example,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Fundamental Theorem

Theorem (Hua 1949 et al.). *Every bijective mapping*

$$\varphi : \mathrm{S}_n(F) \to \mathrm{S}_n(F) : X \mapsto X^\varphi$$

preserving adjacency in both directions is of the form

$$X \mapsto \lambda P X^\sigma P^{\mathrm{T}} + R,$$

where $P \in \mathrm{GL}_n(F)$, $R \in \mathrm{S}_n(F)$, σ is an automorphism of F, and $\lambda \in F \setminus \{0\}$, up to the following exceptional case.
If $F = \mathrm{GF}(2)$ and $n = 3$ then the group of all automorphisms is generated by the transformation $(*)$ and the mappings from above.

The assumptions in Hua's fundamental theorem can be weakened. W.-l. Huang, Höfer, Wan [16]. See also W.-l. Huang [12].

Avoiding Matrices

From a theoretical viewpoint one may define the space of symmetric $n \times n$ matrices over F in a coordinate free way.

with coordinates / matrices	without coordinates / matrices
F^n	V . . . n-dimensional left vector space over F
F^{n*}	V^* . . . dual vector space of V
$\mathrm{S}_n(F)$	space of symmetric bilinear forms on V
	$\cong \mathrm{S}_2(V^*)$ . . . symmetric square of V^*
$a^* \cdot (a^*)^{\mathrm{T}}$	a^*a^* . . . pure symmetric tensor
rank of a matrix	rank of a symmetric bilinear form

Part 4

Symplectic Dual Polar Spaces

We establish an embedding of any space of symmetric matrices in an appropriate symplectic dual polar space.

Symplectic Spaces

Let $\mathrm{PG}(2n-1, F)$ be the projective space over the left vector space F^{2n}, where F is a field.

The matrix

$$K := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

defines a non-degenerate alternating bilinear form

$$F^{2n} \times F^{2n} \to F : (x, y) \mapsto xKy^{\mathrm{T}}.$$

It determines a symplectic polarity on the set of subspaces of $\mathrm{PG}(2n-1, F)$

$$W \mapsto W^{\perp}, \text{ where } W^{\perp} := \{y \in F^{2n} \mid xKy^{\mathrm{T}} = 0 \text{ for all } x \in W\}.$$

We have $\dim W + \dim W^{\perp} = 2n - 2$. (Vector space dimensions sum up to $2n$.)

Subspaces

With respect to $\perp$ any subspace W has precisely one of the following properties.

- *non-isotropic:* W and $W^\perp$ have no point in common.
- *isotropic:* W and $W^\perp$ have at least one common point.
- *totally isotropic:* W is contained in $W^\perp$.

- All points are isotropic. Hence they are also totally isotropic.
- Any line is either non-isotropic or totally isotropic.
- Each totally isotropic subspace is contained in a maximal one. Any maximal totally isotropic subspace W satisfies $W = W^\perp$ and has dimension $n-1$.

Symplectic Polar Spaces

The *symplectic polar space* on $(\mathrm{PG}(2n-1,F),\perp)$ is defined as follows:

- Its points are the points of $\mathrm{PG}(2n-1,F)$.
- Its lines are the totally isotropic lines with respect to $\perp$.

We shall not be concerned with these polar spaces.

Graph on $\mathcal{I}_{2n-1,n-1}(F)$

Let $\mathcal{I}_{2n-1,n-1}(F)$ be the set of all maximal totally isotropic subspaces of $(\mathrm{PG}(2n-1,F),\perp)$. This is a subset of $\mathcal{G}_{2n-1,n-1}(F)$.

- Two totally isotropic $(n-1)$-subspaces W_1 and W_2 are called *adjacent* if

$$\dim W_1 \cap W_2 = n-2.$$

- We consider the set $\mathcal{I}_{2n-1,n-1}(F)$ as the vertices of an undirected graph the edges of which are the (unordered) pairs of adjacent totally isotropic $(n-1)$-subspaces. It is called the *dual polar graph* on $\mathcal{I}_{2n-1,n-1}(F)$.
- Two totally isotropic $(n-1)$-subspaces W_1 and W_2 are at graph theoretical distance $k \geq 0$ if, and only if,

$$\dim W_1 \cap W_2 = n-1-k.$$

- Distance in the dual polar graph = distance in the Grassmann graph.

Symplectic Dual Polar Spaces

The *symplectic dual polar space* on $(\mathrm{PG}(2n-1, F), \perp)$ is defined as follows:

- Its *POINT set* is $\mathcal{I}_{2n-1,n-1}(F)$, i. e., the set of maximal totally isotropic subspaces of $(\mathrm{PG}(2n-1, F), \perp)$.

- Its *LINES* are the pencils of the form

$$\{W \in \mathcal{G}_{2n-1,n-1}(F) \mid U \subset W \subset U^{\perp}\},$$

  where U is any $(n-2)$-dimensional totally isotropic subspace.

Any subspace W in the pencil as above is automatically totally isotropic.

Symplectic Dual . . . (cont.)

The POINTS / LINES of this dual polar space are also POINTS / LINES of the Grassmann space $(\mathcal{G}_{2n-1,n-1}(F), \mathcal{P})$.

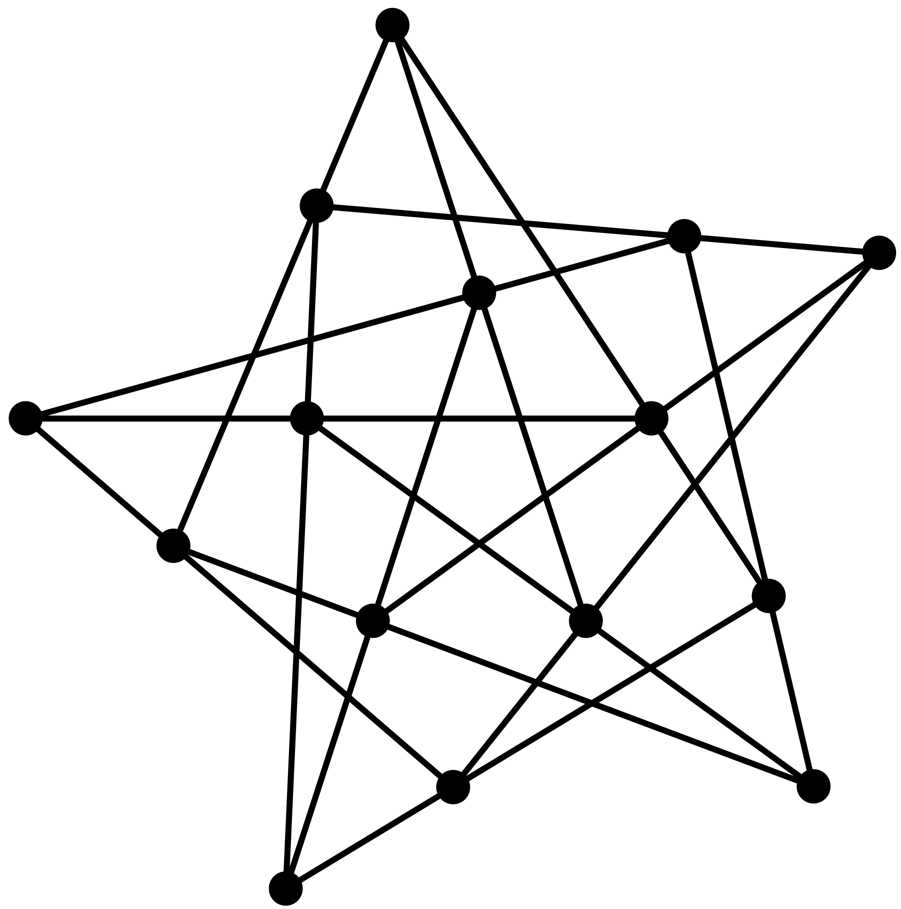
The symplectic dual polar space on $(\mathrm{PG}(2n-1, F), \perp)$ is a connected partial linear space.

Two totally isotropic $(n-1)$-subspaces W_1 and W_2 are adjacent if, and only if, they are distinct and COLLINEAR. In this case the unique LINE joining W_1 and W_2 equals

$$\{W_1, W_2\}^{\sim}.$$

Example

The dual polar space on $(\mathrm{PG}(3, \mathrm{GF}(2)), \perp)$ has 15 POINTS (the 15 lines of a general linear complex in $\mathrm{PG}(2, 3)$) and 15 LINES (the 15 pencils of lines contained in the complex). It coincides with the generalised quadrangle $\mathrm{GQ}(2, 2)$.



We use here the *Cremona-Richmond configuration* for illustration.

Fundamental Theorem

Theorem (W. L. Chow 1949) *Every bijective mapping*

$$\varphi : \mathcal{I}_{2n-1,n-1}(F) \to \mathcal{I}_{2n-1,n-1}(F) : X \mapsto X^{\varphi}$$

preserving adjacency in both directions is of the form

$$X \mapsto \{x^{\sigma}P \mid x \in X \subset F^{2n}\},$$

where $P \in \mathrm{GSp}_{2n}(F)$ and σ is an automorphism of F.

Here GSp_{2n} denotes the general symplectic group: $P \in F^{2n\times 2n}$ is in $\mathrm{GSp}_{2n}(F)$ if, and only if

$$PKP^{\mathrm{T}} = \mu K \quad \text{for some} \quad \mu \in F \setminus \{0\}.$$

The assumptions in Chow's fundamental theorem can be weakened.

W.-l. Huang [13].

An Embedding

We adopt the assumptions from Part 3.

$\mathrm{S}_n(F)$ can be embedded in the Grassmannian $\mathcal{G}_{2n-1,n-1}(F)$ like before:

$$\begin{array}{ccccc} \mathrm{S}_n(F) & \longrightarrow & F^{n\times 2n} & \longrightarrow & \mathcal{G}_{2n-1,n-1}(F) \\ X & \longmapsto & X|I_n & \longmapsto & \text{left rowspace of } X|I_n \end{array}$$

- Matrices $X, Y \in \mathrm{S}_n(F)$ are adjacent if, and only if, their images in $\mathcal{G}_{2n-1,n-1}(F)$ are adjacent.
- LINES of matrices are mapped to LINES (pencils) of the Grassmann space with one element removed.

Projective Matrix Spaces

Let $W \in \mathcal{G}_{2n-1,n-1}(F)$ be an $(n-1)$-subspace with left homogeneous coordinates $(X, Y) \in F^{n\times n} \times F^{n\times n}$. Then the following assertions are equivalent:

1. W is totally isotropic with respect to $\perp$, i. e., $W \in \mathcal{I}_{2n-1,n-1}(F)$.
2. $(X|Y) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} (X|Y)^{\mathrm{T}} = 0$.
3. $XY^{\mathrm{T}} = YX^{\mathrm{T}}$.

In particular, for $Y = I_n$ the last conditions reads $X = X^{\mathrm{T}}$. Hence the embedding from the previous slide can be considered as a mapping

$$\mathrm{S}_n(F) \to \mathcal{I}_{2n-1,n-1}(F).$$

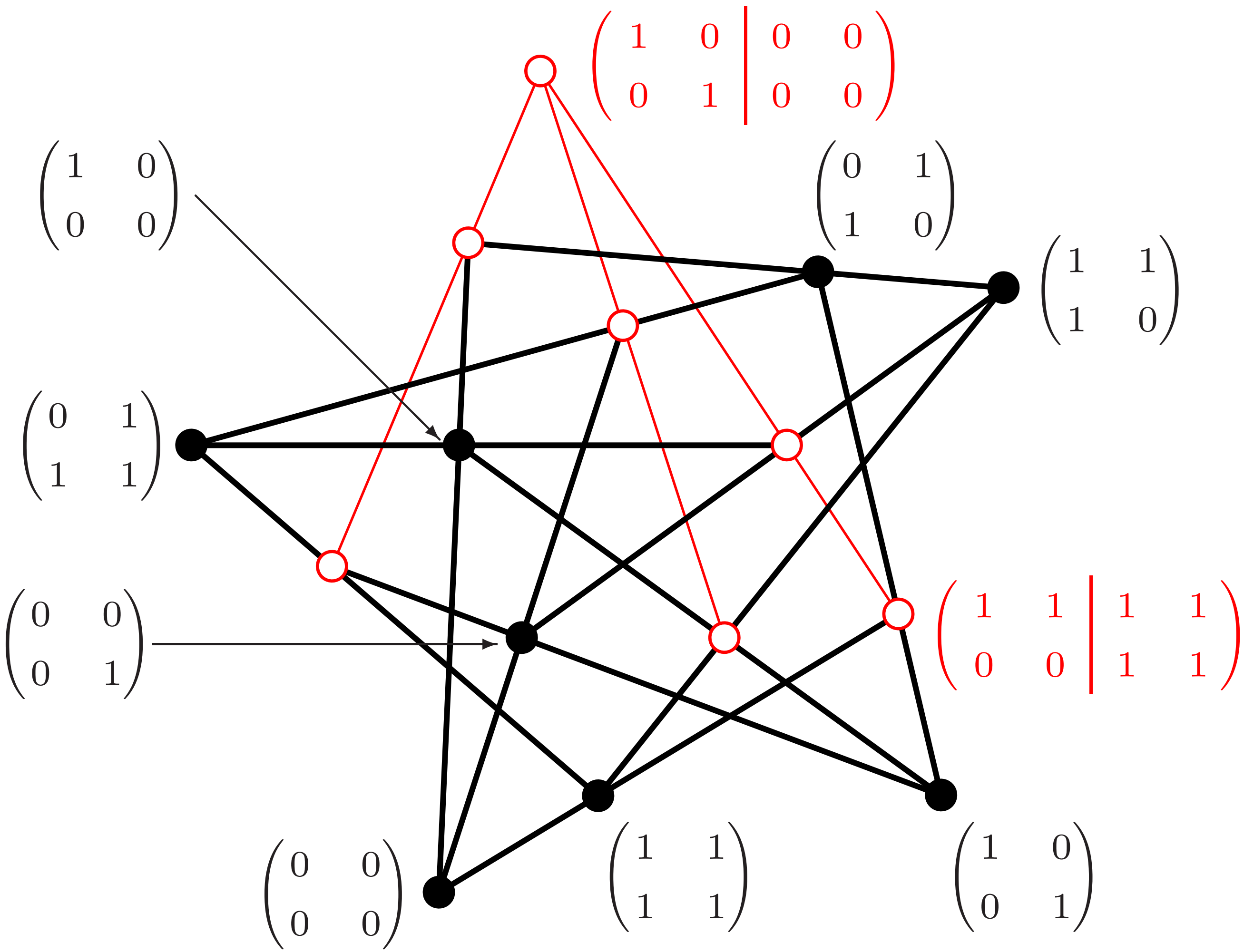
The dual polar space on $\mathcal{I}_{2n-1,n-1}(F)$ is often called the *projective space* of symmetric $n \times n$ matrices over F, even though it is not a projective space in the usual sense.

Points at Infinity

- A totally isotropic subspace with coordinates (X, Y) is in the image of the embedding
$$\mathrm{S}_n(F) \to \mathcal{I}_{2n-1,n-1}(F)$$
if, and only if, Y is invertible. In this case its only preimage is the matrix $Y^{-1}X \in \mathrm{S}_n(F)$.
- All totally isotropic subspaces with coordinates (X, Y), where $Y \notin \mathrm{GL}_m(F)$, are called *points at infinity* of the dual polar space. Clearly, this notion depends on the chosen embedding.
- There is a distinguished totally isotropic subspace of $\mathrm{PG}(2n-1, F)$ given by the left row space of the matrix $(I_n|0)$.
- An element of $\mathcal{I}_{2n-1,n-1}(F)$ is at infinity, precisely when has at least one common point with this $(n-1)$-dimensional subspace.

Example

The space of symmetric 2×2 matrices over $\mathrm{GF}(2)$ can be embedded in the symplectic dual polar space on $(\mathrm{PG}(3, \mathrm{GF}(2)), \perp)$.



Points and lines at infinity are depicted in red.

Jordan Systems

The set $\mathrm{S}_n(F)$ is a Jordan System of the full matrix algebra $R := (F^{n\times n}, +, \cdot)$ over F. Cf. the lecture of A. Blunck or [2].

In terms of our left-homogeneous coordinates $(X, Y) \in R^2$ the POINT set of the "projective space" of symmetric $n \times n$ matrices over F (the set $\mathcal{I}_{2n-1,n-1}(F)$) is the same as the point set of the projective line $\mathbb{P}(\mathrm{S}_n(F))$ over the Jordan system $\mathrm{S}_n(F)$. There is one difference though:

- In the matrix geometric setting the elements of $\mathcal{I}_{2n-1,n-1}(F)$ are characterised by the equation
$$XY^{\mathrm{T}} = YX^{\mathrm{T}}.$$
- In the ring geometric setting the elements of $\mathbb{P}(\mathrm{S}_n(F))$ are given in terms of Bartolone's parametric representation, namely

$$\mathbb{P}(\mathrm{S}_n(F)) = \{R(1 + AB, A) \mid A, B \in \mathrm{S}_n(F)\}.$$

Part 5

Hermitian Matrices

The fifth part deals with some basic notions and results from the Geometry of Hermitian Matrices. Some of the known results depend on technical hypotheses. Here only a brief outline will be given.

Our exposition follows the book of Z.-X. Wan [22], also taking into account recent work.

Basic Notions

- Let F be a field which possesses an *involution*, i. e., an antiautomorphism $\overline{}$ of order two.
- The set $\mathrm{Fix}(\overline{}) =: \mathrm{Fix}$ of fixed elements under $\overline{}$ is closed under addition, but not necessarily closed under multiplication: If $a = \overline{a} \in \mathrm{Fix}$ and $b = \overline{b} \in \mathrm{Fix}$ then $\overline{ab} = \overline{b}\,\overline{a} = ba$ need not coincide with ab.
- We assume that Fix is contained in the centre $Z(F)$, whence it is a subfield of $Z(F)$.
- Let $\mathrm{H}_n(F) \subset F^{n\times n}$, $n \geq 1$, be the set of all Hermitian $n \times n$ matrices over F (with respect to $\overline{}$). So
$$A \in \mathrm{H}_n(F) \quad \Leftrightarrow \quad A = \overline{A}^{\mathrm{T}}.$$
- Hereafter there will always be only one involution $\overline{}$ at the same time. The term Hermitian is always understood with respect the chosen involution.

The set $\mathrm{H}_n(F)$ is a subset of the matrix space $F^{n\times n}$.

Examples

Let $\mathbb{H}$ be the non-commutative field of real quaternions. The centre of $\mathbb{H}$ is the field $\mathbb{R}$ of real numbers.

- The conjugation

$$\mathbb{H} \to \mathbb{H} : x + yi + zj + tk \mapsto x - yi - zj - tk \quad (x, y, z, t \in \mathbb{R})$$

  is an involution of $\mathbb{H}$. It meets the assumption from the previous slide: The set of fixed elements coincides with the centre of $\mathbb{H}$.
- The mapping

$$\mathbb{H} \to \mathbb{H} : x + yi + zj + tk \mapsto x - yi + zj + tk$$

  is an involution of $\mathbb{H}$. It does not meet the assumption from the previous slide: The set of fixed elements equals

$$\{x + zj + tk \mid x, z, t \in \mathbb{R}\},$$

  and is therefore not contained in the centre of $\mathbb{H}$.

Examples (cont.)

Let F be a commutative field and let $^{-}$ be an involution. Then $^{-}$ is an automorphism of F. Moreover, Fix is a subfield of $F = Z(F)$. More precisely, F is a separable quadratic extension of Fix. We mention two examples.

- $F = \mathbb{C}$ and $^{-}$ equals the conjugation

$$\mathbb{C} \to \mathbb{C} : x + yi \mapsto x - yi \quad (x, y \in \mathbb{R}).$$

  Hence $\mathrm{Fix} = \mathbb{R}$.

- $F = \mathrm{GF}(4)$ and $^{-}$ equals the mapping

$$\mathrm{GF}(4) \to \mathrm{GF}(4) : x \mapsto x^2.$$

  Hence $\mathrm{Fix} = \mathrm{GF}(2)$.

  This is the only involution of $\mathrm{GF}(4)$.

A Single Hermitian Matrix

- Each Hermitian matrix $A \in \mathrm{H}_n(F)$ determines a semilinear mapping

$$f_A : F^n \to F^{n*} : y \mapsto A\overline{y}^{\mathrm{T}}.$$

  This provides the link with Part 1. (The dual space F^{n*} can be turned into a left vector space by virtue of $\overline{}$. Then this mapping gets linear, as in Part 1.)

  Moreover, the matrix A defines a Hermitian sesquilinear form

$$g_A : F^n \times F^n \to F : (x, y) \mapsto xA\overline{y}^{\mathrm{T}}.$$

  We shall adopt this interpretation of the matrix A.

- All Hermitian sesquilinear forms $F^n \times F^n \to F$ arise in this way.

Hermitian Rank One Matrices

Given a column vector $a^* = (a_1^*, a_2^*, \ldots, a_m^*)^{\mathrm{T}} \in F^{n*}$ we obtain the Hermitian sesquilinear form

$$F^n \times F^n \to F : (x, y) \mapsto (x \cdot a^*)\overline{(y \cdot a^*)} = x \cdot \left(a^* \cdot (\overline{a}^*)^{\mathrm{T}}\right) \cdot \overline{y}^{\mathrm{T}}.$$

Its matrix is therefore

$$a^* \cdot (\overline{a}^*)^{\mathrm{T}} = \begin{pmatrix} a_1^*\overline{a}_1^* & a_1^*\overline{a}_2^* & \ldots & a_1^*\overline{a}_n^* \\ a_2^*\overline{a}_1^* & a_2^*\overline{a}_2^* & \ldots & a_2^*\overline{a}_n^* \\ \ldots & \ldots & \ldots & \ldots \\ a_n^*\overline{a}_1^* & a_n^*\overline{a}_2^* & \ldots & a_n^*\overline{a}_n^* \end{pmatrix}.$$

This matrix has rank one provided that $a^* \neq 0$. All Hermitian matrices with rank ≤ 1 arise in this way.

Vector Space on $\mathrm{H}_n(F)$

The sum of two Hermitian matrices $A, B \in F^{n\times n}$ corresponds in a natural way to the sum of the associated sesquilinear forms $g_A + g_B$.

For any $\lambda \in \mathrm{Fix}$ the (obviously Hermitian) matrix $\lambda A = A\lambda$ may be viewed as the product of λ and the Hermitan sesquilinear form g_A:

$$(\lambda g_A) : (x, y) \mapsto \lambda(xAy^{\mathrm{T}}) = x(\lambda A)y^{\mathrm{T}}.$$

Hence $\mathrm{H}_n(F)$ is a (left or right) vector space over the commutative field Fix.
Here $\mathrm{Fix} \subset Z(F)$ is essential.

Graph on $\mathrm{H}_n(F)$

We assume $n \geq 2$. Hence $\mathrm{H}_n(F)$ contains matrices of rank ≥ 2.

- The notion of adjacency is inherited form $F^{n\times n}$.
- We consider $\mathrm{H}_n(F)$ as an undirected graph the edges of which are precisely the (unordered) pairs of adjacent Hermitian matrices.
- Two Hermitian matrices A and B are at the graph-theoretical distance $k \geq 0$ if, and only if,

$$\mathrm{rk}(A - B) = k.$$

- The diameter (maximal distance) in this graph is n.

Lines

Given $a^* \in F^{n*} \setminus \{0\}$ and any matrix $R \in \mathrm{H}_n(F)$ the set

$$\{\lambda a^* (\overline{a}^*)^{\mathrm{T}} + R \mid \lambda \in \mathrm{Fix}\}$$

is called a *LINE* of $\mathrm{H}_n(F)$.

Let $\mathcal{L}_{\mathrm{H}}$ be the set of all such LINES. Then $(\mathrm{H}_n(F), \mathcal{L}_{\mathrm{H}})$ is a partial linear space, called the *space of Hermitian $n \times n$ matrices over Fix.*

In this context the elements of $\mathrm{H}_n(F)$ will also be called *POINTS*.

Two Hermitian matrices A and B are adjacent if, and only if, they are distinct and COLLINEAR. In this case the unique LINE joining A and B equals

$$\begin{aligned}\{A, B\}^{\sim} &= \{X \in \mathrm{H}_n(F) \mid (X = A) \text{ or } (X = B) \text{ or } (X \text{ is adjacent to } A \text{ and } B)\} \\ &= \{\lambda(A - B) + B \mid \lambda \in \mathrm{Fix}\}.\end{aligned}$$

Example

We recall that the Galois field $\mathrm{GF}(4) = \{0, 1, \omega, \omega^2\}$ admits a single involution, namely

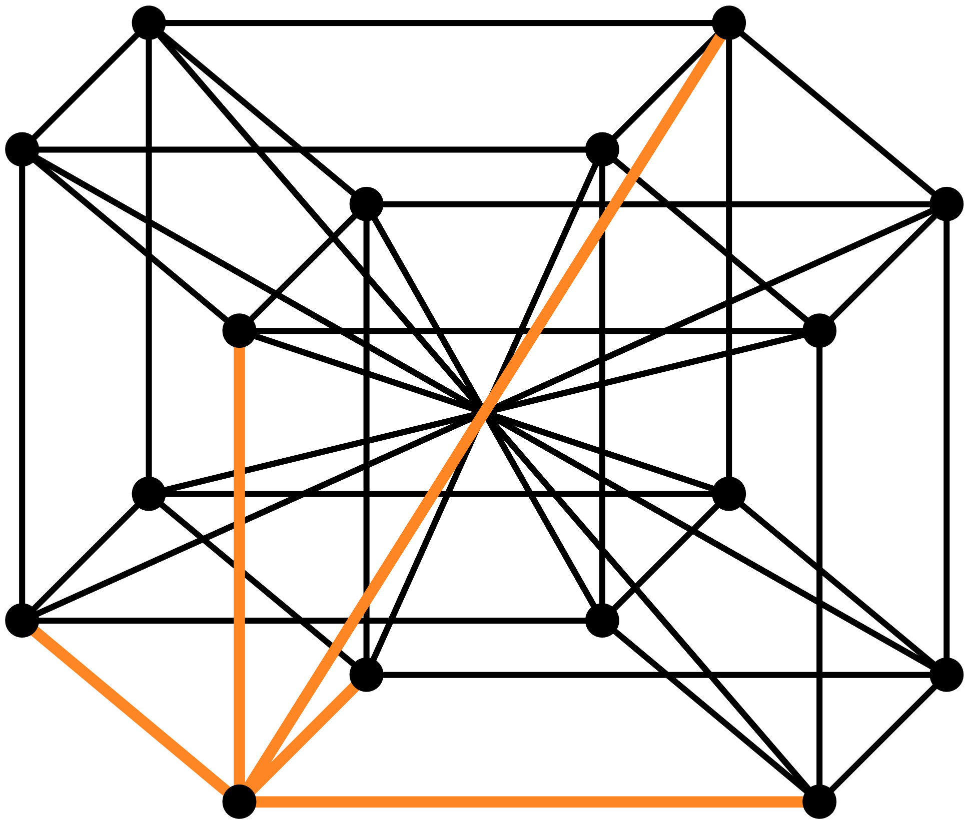
$$\bar{\ } : \mathrm{GF}(4) \to \mathrm{GF}(4) : x \mapsto x^2.$$

The space of Hermitian 2×2 matrices over $\mathrm{GF}(4)$ contains the following five Hermitian matrices with rank 1:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ \omega^2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega^2 \\ \omega & 1 \end{pmatrix}.$$

Thus there are five LINES through the zero matrix, each comprising two POINTS. The space of Hermitian 2×2 matrices over $\mathrm{GF}(4)$ is a *partial affine space*, viz. the affine space on $\mathrm{H}_2(\mathrm{GF}(4))$ over $\mathrm{Fix} = \mathrm{GF}(2)$ with $15 - 5 = 10$ parallel classes of lines removed.

Example (cont.)



The space $\mathrm{H}_2(\mathrm{GF}(4))$ comprises 16 POINTS (matrices) and 40 LINES (of matrices). The five LINES through the zero matrix are depicted in orange. The associated graph is known as the *Clebsch graph*.

Example

We consider the four-dimensional space-time $\mathbb{R}^4$ with the indefinite quadratic form given by the matrix $\mathrm{diag}(1,1,1,-1)$ and the space $\mathrm{H}_2(\mathbb{C})$ with respect to conjugation. The mapping

$$\gamma : \mathbb{R}^4 \to \mathrm{H}_2(\mathbb{C}) : (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_4 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_4 - x_1 \end{pmatrix}$$

is bijective and

$$-\det((x_1, x_2, x_3, x_4)^\gamma) = x_1^2 + x_2^2 + x_3^2 - x_4^2.$$

The light-like lines of $\mathbb{R}^4$ correspond under γ to the LINES of $\mathcal{L}_\mathrm{H}$ and vice versa.

Example

We consider the six-dimensional space-time $\mathbb{R}^6$ with the indefinite quadratic form given by the matrix $\mathrm{diag}(1,1,1,1,1-1)$ and the space $\mathrm{H}_2(\mathbb{H})$ with respect to conjugation. The mapping

$$\gamma : \mathbb{R}^6 \to \mathrm{H}_2(\mathbb{H}) : (x_1, x_2, \ldots, x_6) \mapsto \begin{pmatrix} x_6 + x_1 & x_2 + ix_3 + jx_4 + kx_5 \\ x_2 - ix_3 - jx_4 - kx_5 & x_6 - x_1 \end{pmatrix}$$

is bijective and

$$-\det((x_1, x_2, x_3, x_4)^\gamma) = x_1^2 + x_2^2 + \cdots + x_5^2 - x_6^2.$$

The light-like lines of $\mathbb{R}^6$ correspond under γ to the LINES of $\mathcal{L}_{\mathrm{H}}$ and vice versa.

Summary

- The space $(\mathrm{H}_n(F), \mathcal{L}_{\mathrm{H}})$ is a connected partial linear space.

- The set $\mathrm{H}_n(F)$ can be considered as a (left or right) vector space (affine space) over Fix. The LINES of $\mathcal{L}_{\mathrm{H}}$ comprise some of the parallel classes of lines of this affine space.

Automorphisms

An *automorphism* of the space $(\mathrm{H}_n(F), \mathcal{L}_\mathrm{H})$ is a bijection

$$\varphi : \mathrm{H}_n(F) \to \mathrm{H}_n(F) : X \mapsto X^\varphi$$

preserving adjacency in both directions. Consequently, LINES are mapped onto LINES under φ and φ^{-1}.

Examples

- Translations: $X \mapsto X + R$, where $R \in \mathrm{H}_n(F)$.
- Hermitian congruence transformations: $X \mapsto PX\overline{P}^\mathrm{T}$, where $P \in \mathrm{GL}_n(F)$.
- Field automorphisms: $X \mapsto X^\sigma$, where σ is an automorphism of F commuting with $\overline{}$ and acting on the entries of X.
- Scalings: $X \mapsto \lambda X$, where $\lambda \in \mathrm{Fix} \setminus \{0\}$.
- Transposition: $X \mapsto X^\mathrm{T} = \overline{X}$, but only in certain cases. See next slides.

Transposition

Transposition of any $n \times n$ matrix X over a commutative field preserves the rank. In symbols, we obtain

$$\operatorname{rk} X^{\mathrm{T}} = \operatorname{rk} X \quad \text{for all} \quad X \in F^{n \times n}.$$

Over a commutative field F the mapping $X \to X^{\mathrm{T}}$ deserves no special mention, because the involution $^{-}$ is an automorphism of F and

$$X^{\mathrm{T}} = \overline{X} \quad \text{for all} \quad X \in \mathrm{H}_n(F).$$

Transposition (cont.)

For any $b \in F$ we obtain $\overline{b\overline{b}} = b\overline{b}$. So $b\overline{b} \in \mathrm{Fix} \subset Z(F)$ and, provided that $b \neq 0$,

$$\overline{b}b = (\overline{b}b)(\overline{b}\,\overline{b^{-1}}) = \overline{b}(b\overline{b})\overline{b^{-1}} = (b\overline{b})\overline{b}\,\overline{b^{-1}} = b\overline{b}.$$

For $b = 0$ we clearly have $\overline{b}b = 0 = b\overline{b}$.

Now, given any 2×2 Hermitian matrix, say

$$A = \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix},$$

we notice that $a, c \in \mathrm{Fix}$, whence $a, b, \overline{b}, c$ generate a commutative subfield of F. Hence we can use determinants and obtain

$$\begin{aligned}
\operatorname{rk} A = 2 \quad &\Leftrightarrow \quad \det A \neq 0 \quad \Leftrightarrow \quad \det A^{\mathrm{T}} \neq 0 \quad \Leftrightarrow \quad \operatorname{rk} A^{\mathrm{T}} = 2,\\
\operatorname{rk} A = 1 \quad &\Leftrightarrow \quad \det A = 0 \wedge A \neq 0 \quad \Leftrightarrow \quad \det A^{\mathrm{T}} = 0 \wedge A^{\mathrm{T}} \neq 0 \quad \Leftrightarrow \quad \operatorname{rk} A^{\mathrm{T}} = 1,\\
\operatorname{rk} A = 0 \quad &\Leftrightarrow \quad A = 0 \quad \Leftrightarrow \quad A^{\mathrm{T}} = 0 \quad \Leftrightarrow \quad \operatorname{rk} A^{\mathrm{T}} = 0.
\end{aligned}$$

Transposition (cont.)

Transposition of Hermitian $n \times n$ matrices over a skew field need not preserve the left row rank for $n \geq 3$. For example, over the real quaternions $\mathbb{H}$ we have

$$A := \begin{pmatrix} 1 \\ -i \\ -j \end{pmatrix} \begin{pmatrix} 1 & i & j \end{pmatrix} = \begin{pmatrix} 1 & i & j \\ -i & 1 & -k \\ -j & k & 1 \end{pmatrix}.$$

The transpose of this rank one matrix equals

$$A^{\mathrm{T}} = \overline{A} = \begin{pmatrix} 1 & -i & -j \\ i & 1 & k \\ j & -k & 1 \end{pmatrix}.$$

The matrix A^{T} has (left row) rank two, because the first and second row are linearly independent (from the left).

Fundamental Theorem

Theorem (Hua 1945 et al.) *Under certain hypotheses, every bijective mapping*

$$\varphi : \mathrm{H}_n(F) \to \mathrm{H}_n(F) : X \mapsto X^\varphi$$

preserving adjacency in both directions is of the form

$$X \mapsto \lambda P X^\sigma \overline{P}^{\,\mathrm{T}} + R \qquad \text{or, for } n = 2 \text{ only,} \qquad X \mapsto \lambda P \overline{X}^{\,\sigma} \overline{P}^{\,\mathrm{T}} + R$$

where $P \in \mathrm{GL}_n(F)$, $R \in \mathrm{H}_n(F)$, σ *is an automorphism of* F *commuting with* $\overline{}$, *and* $\lambda \in \mathrm{Fix} \setminus \{0\}$.

See L.-P. Huang and Z.-X. Wan [8] for the case $n = 2$.

The assumptions in Hua's fundamental theorem can be weakened.

W.-l. Huang [14]; W.-l. Huang, R. Höfer, and Z.-X. Wan [16]; W.-l. Huang and P. Šemrl [17].

Part 6

Unitary Dual Polar Spaces

We establish an embedding of any space of Hermitian matrices in an appropriate unitary dual polar space.

Unitary Spaces

Let $\mathrm{PG}(2n-1, F)$ be the projective space over the left vector space F^{2n}, where F is a field. Also let $^{-}$ be a fixed antiautomorphism of F as before.

The matrix

$$K := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

together with $^{-}$ defines a non-degenerate skew-Hermitian sesquilinear form

$$F^{2n} \times F^{2n} \to F : (x, y) \mapsto xK\overline{y}^{\mathrm{T}}.$$

It determines a unitary polarity on the set of subspaces of $\mathrm{PG}(2n-1, F)$

$$W \mapsto W^{\perp}, \text{ where } W^{\perp} := \{y \in F^{2n} \mid xK\overline{y}^{\mathrm{T}} = 0 \text{ for all } x \in X\}.$$

We have $\dim W + \dim W^{\perp} = 2n - 2$. (The vector space dimensions sum up to $2n$.)

Subspaces

With respect to $\perp$ any subspace W has precisely one of the following properties.

- *non-isotropic:* W and $W^\perp$ have no point in common.
- *isotropic:* W and $W^\perp$ have at least one common point.
- *totally isotropic:* W is contained in $W^\perp$.

- There exist totally isotropic and non-isotropic points.
- There exist lines of all three kinds.
- Each totally isotropic subspace is contained in a maximal one. Any maximal totally isotropic subspace W satisfies $W = W^\perp$ and has dimension $n - 1$.

Unitary Polar Spaces

The *unitary polar space* on $(\mathrm{PG}(2n-1,F),\perp)$ is defined as follows:

- Its points are the points of $\mathrm{PG}(2n-1,F)$.
- Its lines are the totally isotropic lines with respect to $\perp$.

We shall not be concerned with these polar spaces.

Graph on $\mathcal{I}_{2n-1,n-1}(F)$

Let $\mathcal{I}_{2n-1,n-1}(F)$ be the set of all maximal totally isotropic subspaces of $(\mathrm{PG}(2n-1,F),\perp)$. This is a subset of $\mathcal{G}_{2n-1,n-1}(F)$.

- Two totally isotropic $(n-1)$-subspaces W_1 and W_2 are called *adjacent* if

$$\dim W_1 \cap W_2 = n-2.$$

- We consider the point set of $\mathcal{I}_{2n-1,n-1}(F)$ as an undirected graph the edges of which are the (unordered) pairs of adjacent totally isotropic $(n-1)$-subspaces. It is called the *dual polar graph* on $\mathcal{I}_{2n-1,n-1}(F)$.
- Two totally isotropic $(n-1)$-subspaces W_1 and W_2 are at graph theoretical distance $k \geq 0$ if, and only if,

$$\dim W_1 \cap W_2 = n-1-k.$$

- Distance in the dual polar graph = distance in the Grassmann graph.

Unitary Dual Polar Spaces

The *unitary dual polar space* on $(\mathrm{PG}(2n-1, F), \perp)$ is defined as follows:

- Its *POINT set* is $\mathcal{I}_{2n-1,n-1}(F)$, i. e., the set of maximal totally isotropic subspaces of $(\mathrm{PG}(2n-1, F), \perp)$.

- Its *LINES* have the form

$$\{W \in \mathcal{I}_{2n-1,n-1}(F) \mid U \subset W \subset U^{\perp}\},$$

  where U is any $(n-2)$-dimensional totally isotropic subspace. So LINES are proper subsets of pencils.

Unitary Dual . . . (cont.)

The POINTS of this dual polar space are also POINTS of the Grassmann space $(\mathcal{G}_{2n-1,n-1}(F), \mathcal{P})$. This does not hold, mutatis mutandis, for LINES of this dual polar space. They are proper subsets of LINES of the ambient Grassmann space.

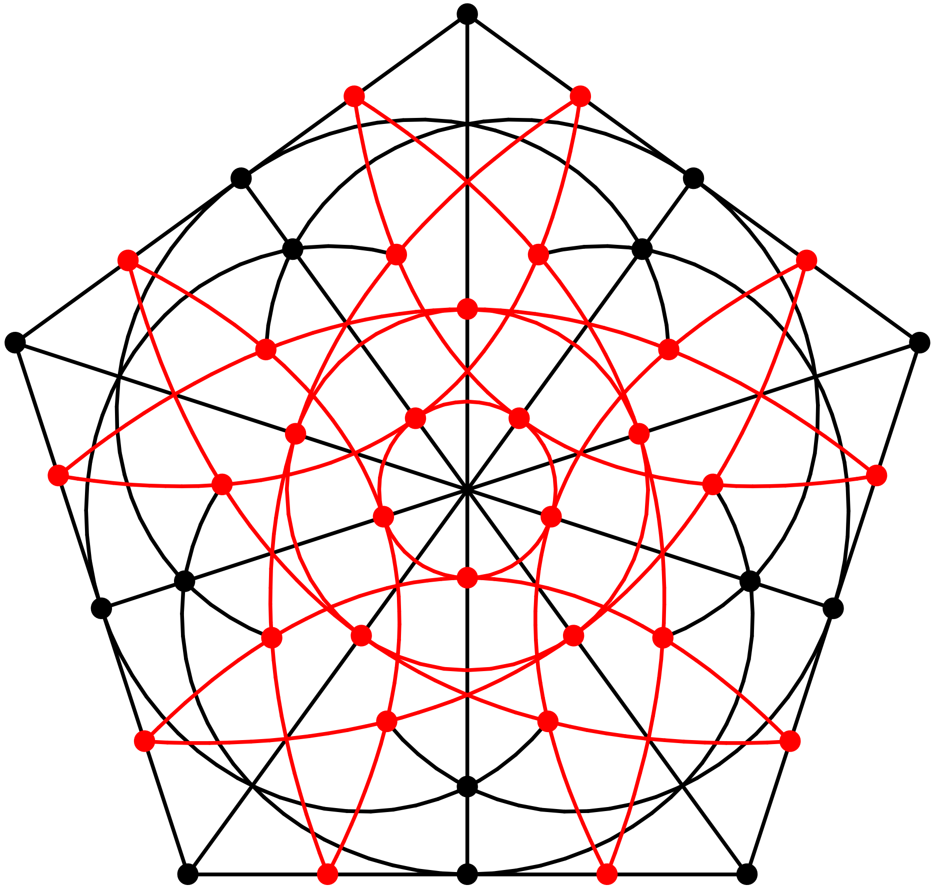
The dual polar space on $(\mathrm{PG}(2n-1, F), \perp)$ is a connected partial linear space.

Two totally isotropic $(n-1)$-subspaces W_1 and W_2 are adjacent if, and only if, they are distinct and COLLINEAR. In this case the unique LINE joining W_1 and W_2 equals

$$\{W_1, W_2\}^{\sim}.$$

Example

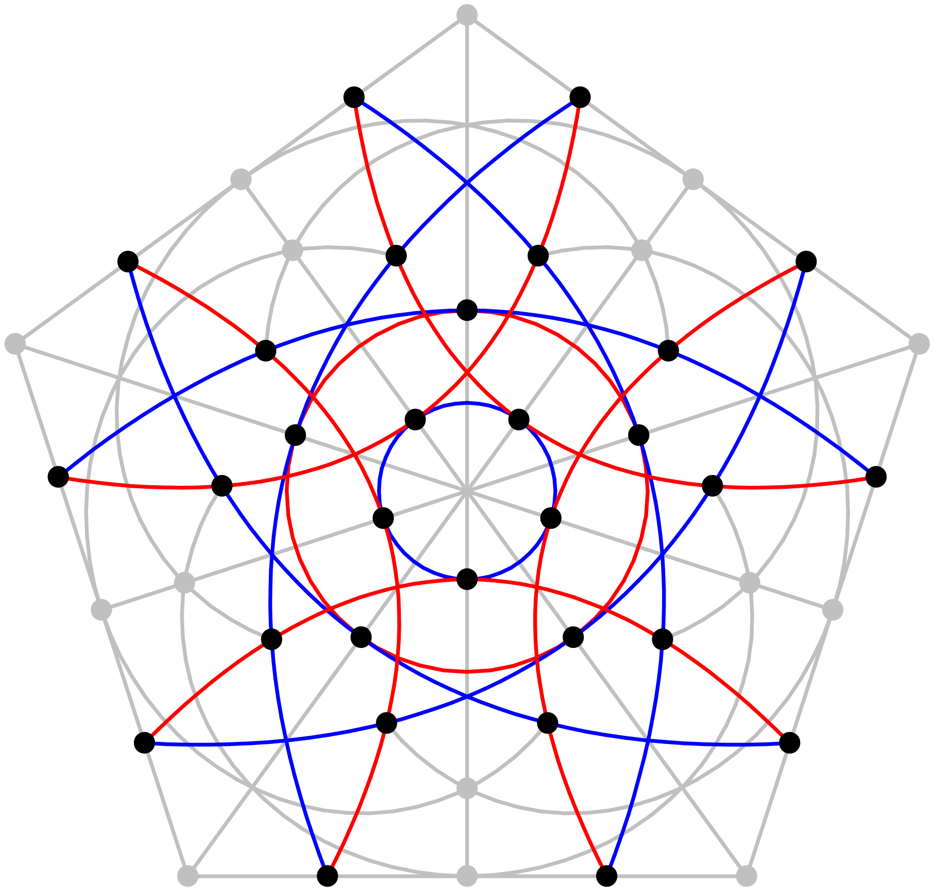
The dual polar space $(\mathrm{PG}(3,4), \perp)$ has 27 POINTS (the 27 totally isotropic lines) and 45 LINES (45 subsets of pencils of lines). It equals the generalised quadrangle $\mathrm{GQ}(2,4)$. We give an illustration of the dual structure. So points / curves below can be viewed as points / lines of $\mathrm{PG}(3,4)$.



The black points and lines constitute a (self-dual) $\mathrm{GQ}(2,2)$.

Example (cont.)

We stick to the terminology from $\mathrm{PG}(3,4)$ and depict the $\mathrm{GQ}(2,2)$ in grey. The remaining $12 = 27 - 15$ totally isotropic lines fall into two classes (red and blue) forming a *double six* of lines: Any two distinct red / blue lines are skew, but each red / blue line meets precisely five of the blue / red lines.



See J. W. P. Hirschfeld [5].

Fundamental Theorem

Theorem (J.-A. Dieudonné 1954 et al.). *Under certain hypotheses, every bijective mapping*

$$\varphi : \mathcal{I}_{2n-1,n-1}(F) \to \mathcal{I}_{2n-1,n-1}(F) : X \mapsto X^{\varphi}$$

preserving adjacency in both directions is of the form

$$X \mapsto \{x^{\sigma} P \mid x \in X \subset F^{2n}\} \quad \text{or, only if } n = 2, \quad X \mapsto \{\overline{x}^{\sigma} P \mid x \in X \subset F^{2n}\}$$

where $P \in \mathrm{GU}_{2n}(F)$ *and* σ *is an automorphism of* F *commuting with* $\overline{}$.

Here GU_{2n} denotes the general unitary group: $P \in F^{2n\times 2n}$ is in $\mathrm{GU}_{2n}(F)$ if, and only if

$$PK\overline{P}^{\mathrm{T}} = \mu K \quad \text{for some} \quad \mu \in F \setminus \{0\}.$$

See also J. Tits [21].

The assumptions in Dieudonné's fundamental theorem can be weakened.

W.-l. Huang [14].

An Embedding

We adopt the assumptions from Part 5.

$\mathrm{H}_n(F)$ can be embedded in the Grassmannian $\mathcal{G}_{2n-1,n-1}(F)$ like before:

$$\begin{array}{ccccc} \mathrm{H}_n(F) & \to & F^{n\times 2n} & \to & \mathcal{G}_{2n-1,n-1}(F) \\ X & \mapsto & X|I_n & \mapsto & \text{left rowspace of } X|I_n \end{array}$$

- Matrices $X, Y \in \mathrm{H}_n(F)$ are adjacent if, and only if, their images in $\mathcal{G}_{2n-1,n-1}(F)$ are adjacent.
- LINES of matrices are mapped to subsets of LINES of the Grassmann space.

Projective Matrix Spaces

Let $W \in \mathcal{G}_{2n-1,n-1}(F)$ be an $(n-1)$-subspace with left homogeneous coordinates $(X, Y) \in F^{n\times n} \times F^{n\times n}$. Then the following assertions are equivalent:

1. W is totally isotropic with respect to $\perp$, i. e., $W \in \mathcal{I}_{2n-1,n-1}(F)$.
2. $(X|Y) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} (\overline{X}|\overline{Y})^{\mathrm{T}} = 0$.
3. $X\overline{Y}^{\mathrm{T}} = Y\overline{X}^{\mathrm{T}}$.

In particular, for $Y = I_n$ the last conditions reads $X = \overline{X}^{\mathrm{T}}$. Hence the embedding from the previous slide can be considered as a mapping

$$\mathrm{H}_n(F) \to \mathcal{I}_{2n-1,n-1}(F).$$

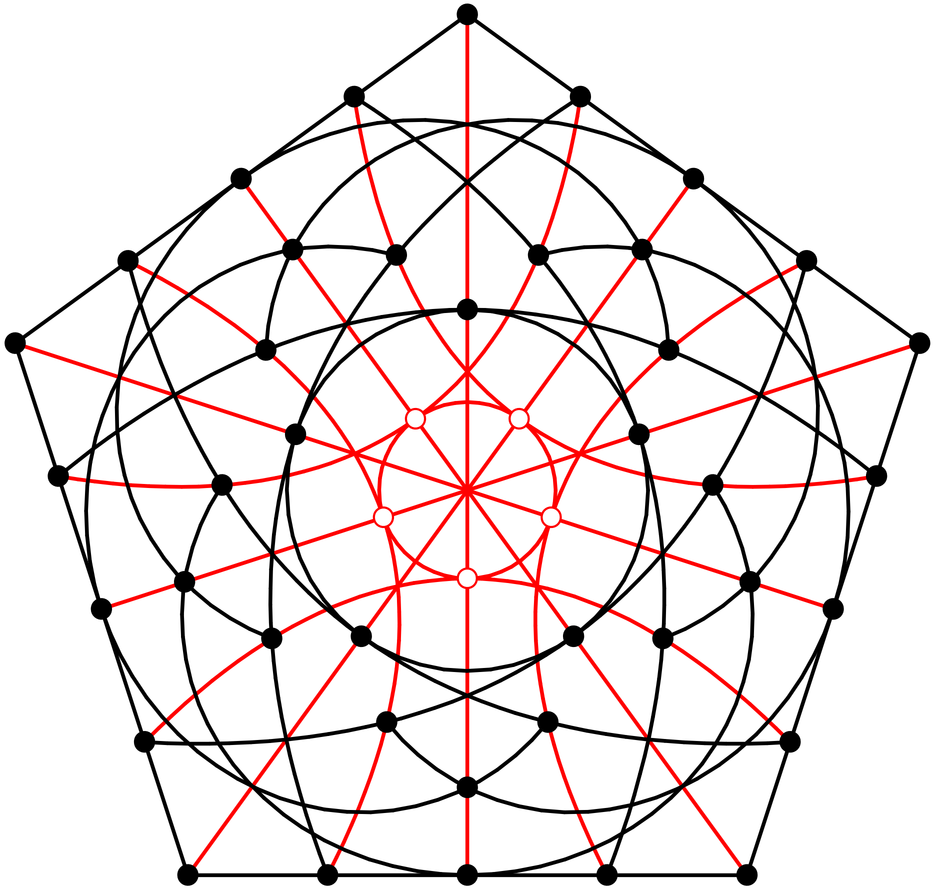
The dual polar space on $\mathcal{I}_{2n-1,n-1}(F)$ is often called the *projective space* of Hermitian $n \times n$ matrices over F, even though it is not a projective space in the usual sense.

Points at Infinity

- A totally isotropic subspace with coordinates (X, Y) is in the image of the embedding
$$\mathrm{H}_n(F) \to \mathcal{I}_{2n-1,n-1}(F)$$
if, and only if, Y is invertible. In this case its only preimage is the matrix $Y^{-1}X \in \mathrm{H}_n(F)$.
- All totally isotropic subspaces with coordinates (X, Y), where $Y \notin \mathrm{GL}_m(F)$, are called *points at infinity* of the dual polar space. Clearly, this notion depends on the chosen embedding.
- There is a distinguished totally isotropic subspace of $\mathrm{PG}(2n-1, F)$ given by the left row space of the matrix $(I_n|0)$.
- An element of $\mathcal{I}_{2n-1,n-1}(F)$ is at infinity, precisely when has at least one common point with this $(n-1)$-dimensional subspace.

Example

The space of Hermitian 2×2 matrices over $\mathrm{GF}(4)$ can be embedded in the dual polar space $(\mathrm{PG}(3, \mathrm{GF}(4)), \perp)$.



As before, we illustrate the dual structures: The black elements depict the dual of the Clebsch graph, the 11 POINTS at infinity are illustrated by red curves.

Jordan Systems

The set $\mathrm{H}_n(F)$ is a Jordan System of the full matrix algebra $R := (F^{n\times n}, +, \cdot)$ over $Z(F)$. Cf. the lecture of A. Blunck or [2].

In terms of our left-homogeneous coordinates $(X, Y) \in R^2$ the POINT set of the "projective space" of Hermitian $n \times n$ matrices over F (the set $\mathcal{I}_{2n-1,n-1}(F)$) is the same as the point set of the projective line $\mathbb{P}(\mathrm{H}_n(F))$ over the Jordan system $\mathrm{H}_n(F)$. There is one difference though:

- In the matrix geometric setting the elements of $\mathcal{I}_{2n-1,n-1}(F)$ are characterised by the equation
$$X\overline{Y}^{\mathrm{T}} = Y\overline{X}^{\mathrm{T}}.$$
- In the ring geometric setting the elements of $\mathbb{P}(\mathrm{H}_n(F))$ are given in terms of Bartolone's parametric representation, namely

$$\mathbb{P}(\mathrm{H}_n(F)) = \{R(1 + AB, A) \mid A, B \in \mathrm{H}_n(F)\}.$$

Part 7

Final Remarks and References

There are several topics which would deserve our attention and a detailed discussion.

Final Remarks

- Spaces of alternating matrices: Adjacency has to be defined differently, since alternating matrices with rank one do not exist [22].
- Orthogonal dual polar spaces: They arise as projective spaces of alternating matrices [22].
- Spaces of block triangular [10], skew-Hermitian matrices [8], and Hermitian matrices with $\bar{\ }$ being more general [7], [9].
- Spaces of matrices over a ring: See [6].
- Polar spaces and dual polar spaces in general [3].
- Analogues of matrix spaces for infinite dimension. Here the approach without coordinates becomes essential.
- Near polygons and their relationship with dual polar spaces.

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The book [22] is equipped with an extensive bibliography covering the relevant literature up to the year 1996. See [20] for a more recent survey.