An Introduction to Geometry over Rings

Alexander Kreuzer

ZiF Cooperation Group
Finite Projective Ring Geometries:
An Intriguing Emerging Link Between Quantum Information Theory, Black-Hole Physics and Chemistry of Coupling

Bielefeld, October 2009

Contents

Section 1

The Beginning

J. Hjelmslev

Between 1916 and 1949 J. Hjelmslev discussed in 8 papers a natural geometry

Papers:
Die Geometrie der Wirklichkeit (Acta Math 40 (1916))
Die natürliche Geometrie (Abh. Math Sem. Univ Hamburg 2 (1923))
Einleitung in die allgemeine Kongruenzlehre I - VI (Danske Vid. Selek. Math (1929-1949))

Main ideas

His main ideas:

- For any line there is in every point exactly one perpendicular line.
- In a quadrangle with three right angle also the fourth angle is right.
- Two distinct line may intersect in a line segment.
- Some points may have more then one connecting lines.

neighbours

We call two points neighbours if there is more then one connecting line.

We call two lines G, H neighbours if for any point $x \in G$ there is a neighbour point $y \in H$ and vice versa.

Properties

Hjelmslev:
The relation neighbour should be an equivalence relation

The identification of neighbour points and neighbour lines should result in a known Großgeometrie

Using the dual numbers $H = \mathbb{R} + \mathbb{R}\epsilon$ as coordinates he considered 1929 an example with such properties.

Dual numbers

For a field $\mathbb{K}$ let $\quad H := \mathbb{K} + \mathbb{K}\epsilon$

with $\epsilon \cdot \epsilon = 0$

hence for $\mathbb{K} = \mathbb{R}$
$(2 + 4\epsilon) \cdot (5 + 3\epsilon) \ = \ 10 + (6 + 20)\epsilon = 10 + 26\epsilon$

For $\mathbb{K} = \mathbb{Z}_2$ and $H_2 = \mathbb{Z}_2 + \mathbb{Z}_2\epsilon$:
$\mathbb{Z}_2 = \{0,\ 1\}$
$H_2 = \{0,\ \epsilon,\ 1,\ 1 + \epsilon\}$

affine plane over K

We consider the affine plane $AG(2, K)$ over K.

Points: $A' := K \times K$.

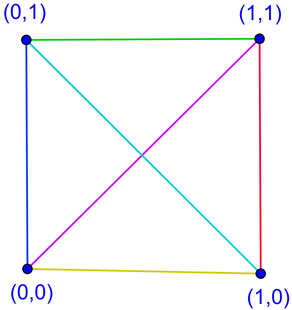
Lines: $\mathfrak{L}' := \{a + Kb \ : \ a, b \in A' \text{ and } b \neq (0, 0)\}$

affine plane over $\mathbb{Z}_2$

We consider the affine plane $AG(2, \mathbb{Z}_2)$ over $\mathbb{Z}_2$

Points: $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0)(1,1)\}$

Lines: $\{\ \mathbb{Z}_2(1,0), \quad (0,1) + \mathbb{Z}_2(1,0),$
$\mathbb{Z}_2(0,1), \quad (1,0) + \mathbb{Z}_2(0,1)$
$\mathbb{Z}_2(1,1), \quad (1,0) + \mathbb{Z}_2(1,1)\}$



Hjelmslev plane over H_2

We consider the affine Hjelmslev plane $AHG(2, H_2)$ over H_2

Points: $A := H_2 \times H_2$

the following mapping identify neighbouring points:

$$\pi : \quad \begin{array}{ccc} H_2 \times H_2 & \rightarrow & \mathbb{Z}_2 \times \mathbb{Z}_2 \\ (x + x'\epsilon, y + y'\epsilon) & \rightarrow & (x, y) \end{array}$$

Hjelmslev plane over H_2

affine plane	Hjelmslev plane
point: (0,0)	points: $(0,0), (\epsilon, 0), (0, \epsilon), (\epsilon, \epsilon)$
point: (0,1)	points: $(0,1), (\epsilon, 1), (0, 1+\epsilon), (\epsilon, 1+\epsilon)$
line: $\mathbb{Z}_2(0,1)$	lines: $H_2(0,1)$, $(\epsilon,0)+H_2(0,1)$ $H_2(\epsilon,1)$, $(\epsilon,0)+H_2(\epsilon,1)$

.

Definition

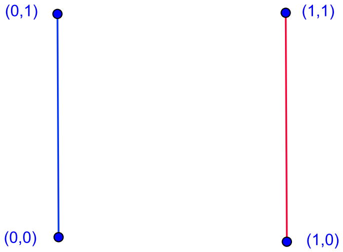
Therefore we define the affine Hjelmslev plane $AHG(2, H_2)$ over H_2:

Points: $P = H_2 \times H_2$

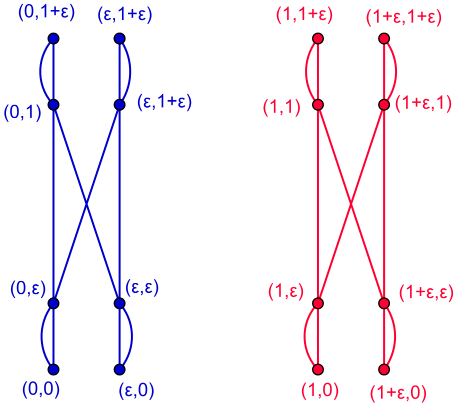
Lines: $\mathfrak{L} := \{a + H_2 b : a, b \in P \quad \text{and} \quad \pi(b) \neq (0,0)\}$

Hjelmslev plane over H_2

affine plane

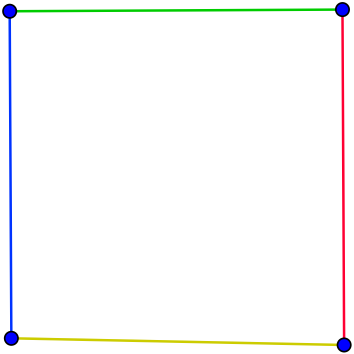


Hjelmslev plane

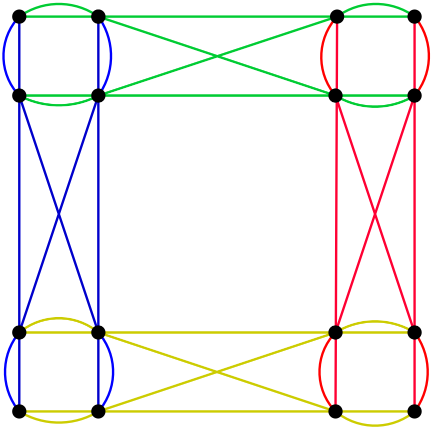


Hjelmslev plane over H_2

affine plane

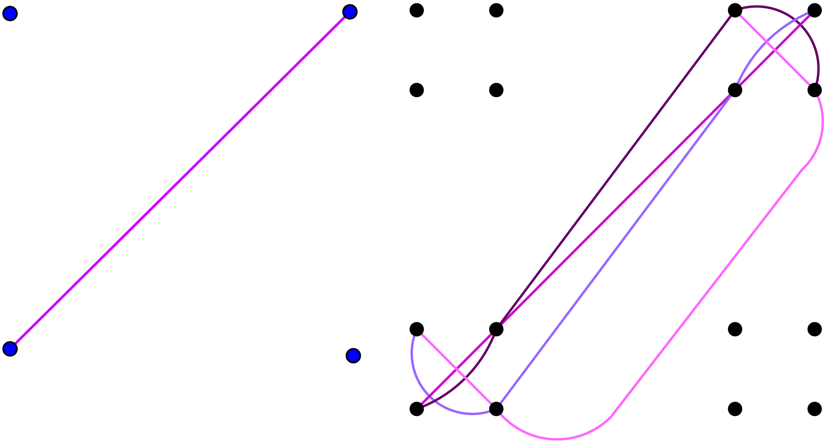


Hjelmslev plane



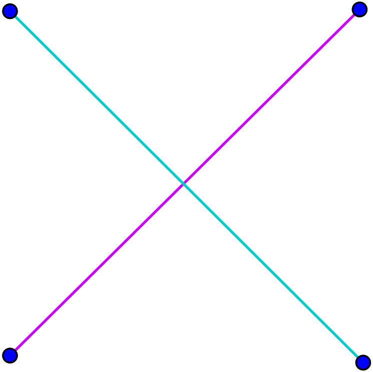
Hjelmslev plane over H_2

affine plane — Hjelmslev plane

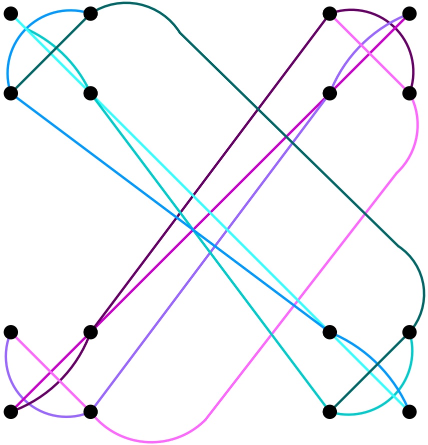


Hjelmslev plane over H_2

affine plane



Hjelmslev plane



affine Hjelmslev plane over H_2

We have:

16 points

24 lines

6 parallel classes

every parallel class contains 4 lines

the parallel classes are not uniquely determined

further papers

There are other papers with deal with planes over rings:

D. Barbilian 1940

G.J. Everett 1942

Section 2

Projective and affine Hjelmslev Planes

projective Hjelmslev planes

1954 W. Klingenberg act on the suggestions of Hjelmslev.
He defines projective „Ebenen mit Nachbarelementen “

Today the notion „projective Hjelmslev planes “ is common.

characteristic properties

The characteristic properties are

- Any two points have a connecting line
- The relation „neighbour “is defined geometrically
- Points which are not neighbours have a unique connecting line
- Identifying neighbouring points leads to a projective (affine) plane.

Incidence structure

Let P denote a set of points
Let $\mathfrak{L}$ be a subset of the power set $\mathfrak{P}(P)$
The elements of $\mathfrak{L}$ are called lines.

The pair $(P, \mathfrak{L})$ is called incidence structure
We say a point $x \in P$ is incident with a line $L \in \mathfrak{L}$, if $x \in L$.

Incidence structure

Let P denote a set of points
Let $\mathfrak{L}$ be a subset of the power set $\mathfrak{P}(P)$
The elements of $\mathfrak{L}$ are called lines.

The pair $(P, \mathfrak{L})$ is called incidence structure
We say a point $x \in P$ is incident with a line $L \in \mathfrak{L}$, if $x \in L$.

If
(I1) Any two points are incident with exactly one line

(I2) Any line contains at least two point

we call $(P, \mathfrak{L})$ a linear space or an incidence space.

projective plane

An incidence space $(P, \mathfrak{L})$ is a projective plane, if

(I3) $|L| \geq 3$ for any line $L \in \mathfrak{L}$

(I4) $|L \cap G| \neq \emptyset$ for any lines $L, G \in \mathfrak{L}$.

projective Hjelmslev plane

Let $(P, \mathfrak{L})$ be an incidence structure. We define a relation neighbouring on the point and line set.

(D1) $a, b \in P$ are neighbours, if there are distinct lines $L, G \in \mathfrak{L}$ with $a, b \in L, G$.

(D2) $G, L \in \mathfrak{L}$ are neighbours, if $|G \cap L| \geq 2$.

We write $a \approx b$ and $G \approx H$

projective Hjelmslev plane

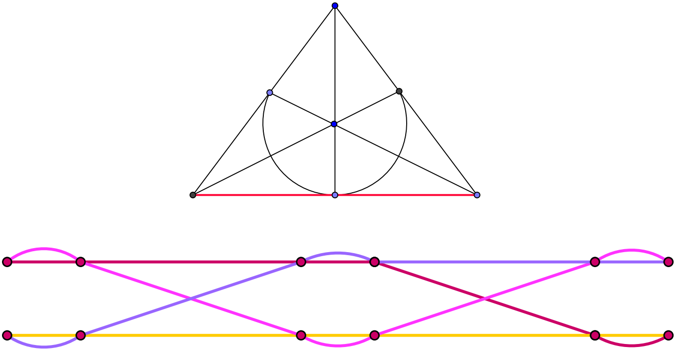
An incidence structure $(P, \mathfrak{L})$ is called a projective Hjelmslev plane, if

(PH1) For all distinct points $a, b \in P$ there exists a line $L \in \mathfrak{L}$ with $a, b \in L$

(PH2) $|L \cap G| \neq \emptyset$ for any lines $L, G \in \mathfrak{L}$

(PH3) There is an epimorphism π on a projective plane $(\pi P, \pi\mathfrak{L})$ with
(i) $a \approx b \Leftrightarrow \pi(a) = \pi(b)$ (ii) $G \approx L \Leftrightarrow \pi(G) = \pi(L)$

example



affine derivation

Let $(P, \mathfrak{L})$ be a projective Hjelmslev plane. Let $F \in \mathfrak{L}$.

$[F] := \{L \in \mathfrak{L} : \; L \approx F\}$ the neighbour class of F.

$A := P \setminus \left(\bigcup_{L \in [F]} L\right)$

$\mathfrak{L}_A := \{L' = L \cap A : L \in \mathfrak{L} \setminus [F]\}$

$L' \parallel G' \; :\Leftrightarrow \; L \cap G \in F$

$(A, \mathfrak{L}_A \parallel)$ is called the affine derivation.

affine Hjelmslev plane

1962 H. Lüneburg uses properties of the affine derivation as axioms for an affine Hjelmslev plane.

(D2') $G, L \in \mathfrak{L}$ are neighbours if for every $a \in G$ there is a neighbour point $b \in L$ and vice versa.

An incidence structure $(A, \mathfrak{L}, \|)$ with an equivalence relation „parallel "is called an affine Hjelmslev plane, if

affine Hjelmslev plane

(AH1) For all distinct points $a, b \in A$ there exists a line $L \in \mathfrak{L}$ with $a, b \in L$

(AH2) For every point $a \in A$ and every line $L \in \mathfrak{L}$ there is a unique line $G \in \mathfrak{L}$ with $a \in G$ and $G \parallel L$

(AH3) For $G \cap L \neq \emptyset$ it holds $G \approx L \Leftrightarrow |G \cap L| \geq 2$

(AH4) There is an epimorphism π on an affine plane $(\pi A, \pi \mathfrak{L})$ with

(i) $a \approx b \Leftrightarrow \pi(a) = \pi(b)$ (ii) $G \approx L \Leftrightarrow \pi(G) = \pi(L)$
(iii) $G \cap L = \emptyset \Rightarrow \pi(G) \parallel \pi(L)$

Questions

- What properties are necessary to introduce special rings (H-rings, AH-rings) as coordinates in affine/projective Hjelmslev planes?
- Is it possible to complete an affine Hjelmslev plane to a projective Hjelmslev plane?
- What are characteristic numbers for finite Hjelmslev planes?

Section 3

Desarguesian Hjelmslev planes and Hjelmslev rings

Translation

Let $(A, \mathfrak{L}, \parallel)$ be an affine Hjelmslev plane.

A bijective mapping $\sigma : A \to A$ is a dilatation , if $\sigma(L) \parallel L$ for a line $L \in \mathfrak{L}$.

A dilatation τ is called quasi translation, if $\tau = id$ or $\tau(x) \neq x$ for all $x \in A$.

Translation

Let $(A, \mathfrak{L}, \parallel)$ be an affine Hjelmslev plane.

A bijective mapping $\sigma : A \to A$ is a dilatation , if $\sigma(L) \parallel L$ for a line $L \in \mathfrak{L}$.

A dilatation τ is called quasi translation, if $\tau = id$ or $\tau(x) \neq x$ for all $x \in A$.

A quasi translation τ is called translation, if for $\tau(F) = F$ it follows that also $\tau(L) = L$ (trace) for any parallel line L of F.

affine axioms of Desargues

(δ) All translation form a group which operates transitively on the point set.

W. Seier 74, PH., Y. Bacon 74:
(δ) The translations operate transitively on the point set.

affine axioms of Desargues

(δ) All translation form a group which operates transitively on the point set.

W. Seier 74, PH., Y. Bacon 74:
(δ) The translations operate transitively on the point set.

(Δ) If any trace of a translation τ is also a trace of a translation τ', then there exists an epimorphism of the translation group, which maps τ on τ'.

affine axioms of Desargues

(δ) All translation form a group which operates transitively on the point set.

W. Seier 74, PH., Y. Bacon 74:
(δ) The translations operate transitively on the point set.

(Δ) If any trace of a translation τ is also a trace of a translation τ', then there exists an epimorphism of the translation group, which maps τ on τ'.

(Δ') Let τ be translation with $\tau(x) \not\approx x$. If any trace of a translation τ is also a trace of a translation τ', then there exists an epimorphism of the translation group, which maps τ on τ'.

H-ring and AH-ring

Let R be an associative ring with $1 \neq 0$. R is called an affine Hjelmslev ring (AH-ring) if:

(HR1) Every element is either unit or zero divisor.

(HR2) All zero divisors are two sided and the set N of all zero divisor is a two sided ideal.

(HR3) For $a, b \in N$ there holds $a \in bH$ or $b \in aH$.

H-ring and AH-ring

Let R be an associative ring with $1 \neq 0$. R is called an affine Hjelmslev ring (AH-ring) if:

(HR1) Every element is either unit or zero divisor.

(HR2) All zero divisors are two sided and the set N of all zero divisor is a two sided ideal.

(HR3) For $a, b \in N$ there holds $a \in bH$ or $b \in aH$.

An AH-ring R is a Hjelmslev ring (H-ring), if

(HR4) For $a, b \in N$ there holds $a \in Hb$ or $b \in Ha$

local ring

Remark: An AH-ring or an H-ring is always a local ring,
i.e., there is a unique maximal ideal.

Examples are the dual numbers or the factor ring $K[x]/(x^n)$ for any commutative field K and any $n \in \mathbb{N}$

There are AH-rings which are not H-rings.

example

R.Baer 42, L.A. Skorniakov 64, Bacon 74:
Let K be a commutative field and ρ an isomorphism on a proper subfield $F \leq K$

For example $K = \mathbb{R}((t)),\ F = \mathbb{R}((t^2)),\ \ \rho : \Sigma\alpha_i t^i \to \Sigma\alpha_i t^{2i}$

example

R.Baer 42, L.A. Skorniakov 64, Bacon 74:
Let K be a commutative field and ρ an isomorphism on a proper subfield $F \leq K$

For example $K = \mathbb{R}((t)),\ F = \mathbb{R}((t^2)),\ \ \rho : \Sigma\alpha_i t^i \to \Sigma\alpha_i t^{2i}$

$$R := K \times K, \quad (a_1, a_2) + (b_1, b_2) := (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) := (a_1 \cdot b_1, a_1 \cdot b_2 + a_2 \cdot \rho(b_1))$$

R is an AH-ring, but not an H-ring.

$$(0, t) \cdot (x_1, x_2) = (0, t \cdot \rho(x_1)) \neq (0, 1) \quad \Rightarrow (0, 1) \notin (0, t)R$$
$$(0, 1) \cdot (x_1, x_2) = (0, \rho(x_1)) \neq (0, t) \quad \Rightarrow (0, t) \notin (0, 1)R$$

A(R)

Let R be an AH-ring or an H-ring.

$$A := R \times R$$

$$\mathfrak{L} := \{(a_1, a_2) + R(b_1, b_2) \mid (a_1, a_2), (b_1, b_2) \in A \text{ and } (b_1, b_2) \notin N \times N\}$$

$$(a_1, a_2) + R(b_1, b_2) \parallel (c_1, c_2) + R(d_1, d_2) \iff R(b_1, b_2) = R(d_1, d_2)$$

$A(R) =: (A, \mathfrak{L}, \parallel)$ (affine Hjelmslev coordinate plane)

Theorem

W.Klingenberg 55, J.W. Lorimer and N.D. Lane 75:

THEOREM: (i) Let R be an AH-ring. Then $A(R)$ is an affine Hjelmslev plane with (δ) and (Δ').

(ii) Let R be an H-ring. Then $A(R)$ is an affine Hjelmslev plane with (δ) and (Δ).

Theorem

W.Klingenberg 55, J.W. Lorimer and N.D. Lane 75:

THEOREM: (i) Let R be an AH-ring. Then $A(R)$ is an affine Hjelmslev plane with (δ) and (Δ').

(ii) Let R be an H-ring. Then $A(R)$ is an affine Hjelmslev plane with (δ) and (Δ).

Let $(A, \mathfrak{L}, \|)$ be an affine Hjelmslev plane.

(iii) If (δ) and (Δ') are valid, then there exists an AH-ring R and a parallelism preserving isomorphism from $A(R)$ to $(A, \mathfrak{L}, \|)$.

(iv) If (δ) and (Δ) are valid, then there exists an H-ring R and a parallelism preserving isomorphism from $A(R)$ to $(A, \mathfrak{L}, \|)$.

$\Pi(R)$

Let R be an H-ring.

$$P := \{x = R(x_1, x_2, x_3) \mid (x_1, x_2, x_3) \in R \times R \times R \setminus N \times N \times N\}$$

$$\mathfrak{L} := \{L = (a_1, a_2, a_3)R \mid (a_1, a_2, a_3) \in R \times R \times R \setminus N \times N \times N\}$$

$$x = R(x_1, x_2, x_3) \in L = (a_1, a_2, a_3)R \iff x_1 a_1 + x_2 a_2 + x_3 a_3 = 0$$

$\Pi(R) := (P, \mathfrak{L})$ (projective Hjelmslev coordinate plane)

projective H-plane

Remark: For an H-ring R, $\Pi(R)$ is a projective Hjelmslev plane.

For an AH-ring R, in general, $\Pi(R)$ is a not projective Hjelmslev plane.

We call a projective Hjelmslev plane $(P, \mathfrak{L},)$ desarguesian , if there is an H-ring R such that $\Pi(R)$ is isomorphic to $(P, \mathfrak{L})$.

projective H-plane

Remark: For an H-ring R, $\Pi(R)$ is a projective Hjelmslev plane.

For an AH-ring R, in general, $\Pi(R)$ is a not projective Hjelmslev plane.

We call a projective Hjelmslev plane $(P, \mathfrak{L},)$ desarguesian , if there is an H-ring R such that $\Pi(R)$ is isomorphic to $(P, \mathfrak{L})$.

There are many papers which characterize desarguesian projective Hjelmslev planes by figures and/or by properties of the contained affine Hjelmslev planes (at least three are necessary).

affine $\longleftrightarrow$ projective

IIn general we cannot complete an affine Hjelmslev plane $A(R)$ over an AH-ring R (i.g.) to a projective Hjelmslev plane.

In general an affine Hjelmslev plane cannot be be completed to a projective Hjelmslev plane.

affine $\longleftrightarrow$ projective

IIn general we cannot complete an affine Hjelmslev plane $A(R)$ over an AH-ring R (i.g.) to a projective Hjelmslev plane.

In general an affine Hjelmslev plane cannot be be completed to a projective Hjelmslev plane.

If (δ) and (Δ) is valid, then an affine Hjelmslev plane can be completed to a projective Hjelmslev plane.

local rings

Klingenberg 56, Bacon 76ff, Machala 75, Baker, Lane, Lorimer 88ff:

As a generalization of Hjelmslev planes, we consider the planes $A(R)$ and $\Pi(R)$ for a local ring R:

local rings

Klingenberg 56, Bacon 76ff, Machala 75, Baker, Lane, Lorimer 88ff:

As a generalization of Hjelmslev planes, we consider the planes $A(R)$ and $\Pi(R)$ for a local ring R:

Characteristic properties:

- The relation „neighbour “is a given equivalence relation
- In general, any two points have no connecting line
- Points which are not neighbours have a unique connecting line
- Identifying neighbouring points leads to a projective (affine) plane

projective Klingenberg plane

Planes over local rings are sometimes called Klingenberg planes, or also Hjelmslev planes, or planes over local rings.

projective Klingenberg plane

Planes over local rings are sometimes called Klingenberg planes, or also Hjelmslev planes, or planes over local rings.

An incidence structure $(P, \mathfrak{L}, \approx, \approx)$ with equivalence relations „neighbour", $\approx$, on the point set P and the line set $\mathfrak{L}$, is called a projective Klingenberg plane, if

(PK1) For all not neighbouring points $a, b \in P$ there exists a unique line $L \in \mathfrak{L}$ with $a, b \in L$.

(PK2) $|L \cap G| \neq \emptyset$ for any not neighbouring lines $L, G \in \mathfrak{L}$.

(PK3) There is an epimorphism π on a projective plane $(\pi P, \pi\mathfrak{L})$ with

(i) $a \approx b \Leftrightarrow \pi(a) = \pi(b)$ (ii) $G \approx L \Leftrightarrow \pi(G) = \pi(L)$

affine Klingenberg plane

An incidence structure $(A, \mathfrak{L}, \approx, \approx, \|)$ with equivalence relations „neighbour“ on the point set and the line set, and an equivalence relation „parallel“ on the point set, is called an affine Klingenberg plane, if,

(AK1) For all not neighbouring points $a, b \in A$ there exists a unique line $L \in \mathfrak{L}$ with $a, b \in L$.

(AK2) For all points $a \in A$ and all lines $L \in \mathfrak{L}$ there is a unique line $G \in \mathfrak{L}$ with $a \in G$ and $G \parallel L$.

(AK3) There is an epimorphism π on an affine plane $(\pi A, \pi\mathfrak{L})$ with

(i) $a \approx b \Leftrightarrow \pi(a) = \pi(b)$
(ii) $G \approx L \Leftrightarrow \pi(G) = \pi(L)$
(iii) $G \cap L = \emptyset \Rightarrow \pi(G) \parallel \pi(L)$
(iv) $\mid \pi(G) \cap \pi(L) \mid = 1 \Rightarrow \mid G \cap L \mid = 1$

Axiom of Desargues

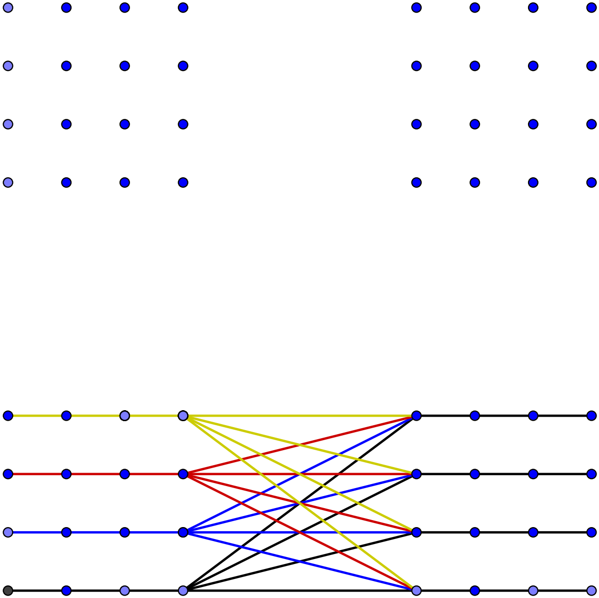
We call a projective Klingenberg planes $(P, \mathfrak{L}, \approx, \approx)$ desarguesian, if there is a local ring R with $(P, \mathfrak{L}, \approx, \approx) \cong \Pi(R)$.

We call an affine Klingenberg planes $(A, \mathfrak{L}, \approx, \approx, \|)$ desarguesian, if there is a local ring R with $(A, \mathfrak{L}, \approx, \approx, \|) \cong A(R)$.

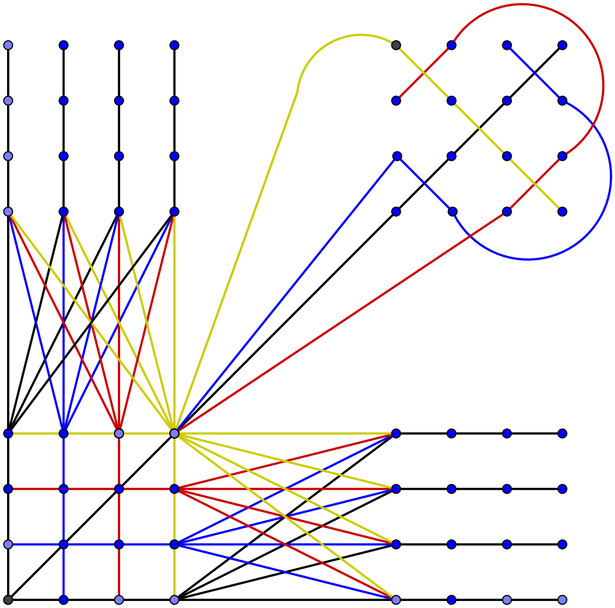
There are a lot of papers on a characterization /geometric figures of the axiom of Desargues

Bacon 76,79,83, Machala 75, Mäurer/Nolte 86, Baker/Lane/Lorimer 88

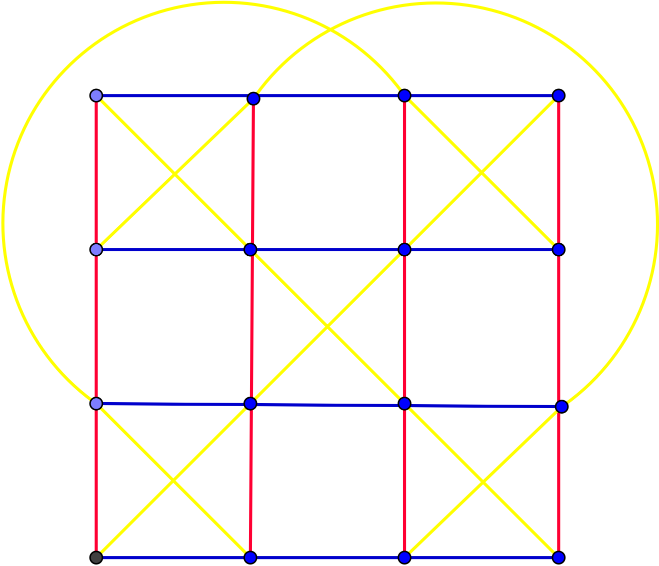
Klingenberg plane



Klingenberg plane



Klingenberg plane



Section 4

Embedding of affine Hjelmslev planes

Question

Is it possible to embed any (desarguesian) AH-plane into a projective PH-plane?

If yes, is the PH-plane unique (up to isomorphism)?

uniform Hjelmslev plane

Kleinfeld 1959, Lüneburg 1962:

We call an AH-plane or PH-plane, respectively, uniform if for any two lines $L, G \in \mathfrak{L}$:

If $G \approx L$, $a, b, \in G, a \in L$ and $a \approx b$, then $b \in L$.

uniform Hjelmslev plane

Kleinfeld 1959, Lüneburg 1962:

We call an AH-plane or PH-plane, respectively, uniform if for any two lines $L, G \in \mathfrak{L}$:

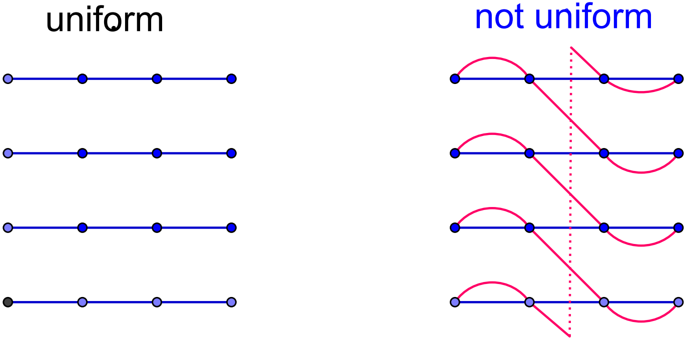
If $G \approx L$, $a, b, \in G, a \in L$ and $a \approx b$, then $b \in L$.

$(A, \mathfrak{L})$ is uniform if and only if the restriction to a neighbour class of any point $([a], \mathfrak{L}([a])$ is an affine plane.

$$[a] := \{b \in A \;:\; b \approx a\}$$

$$\mathfrak{L}([a]) := \{L \cap [a] \;:\; L \in \mathfrak{L} \text{ and } L \cap [a] \neq \emptyset\}$$

figure



Theorems

Drake 68:
Any finite uniform AH-plane can be embedded into a uniform PH-plane.

BUT: There are uniform AH-planes witch are not embeddable in a uniform PH-plane!
(using AH-rings which are not H-rings:)

Theorems

Drake 68:
Any finite uniform AH-plane can be embedded into a uniform PH-plane.

BUT: There are uniform AH-planes witch are not embeddable in a uniform PH-plane!
(using AH-rings which are not H-rings:)

Drake 68, Dembowski 68:
Strongly desarguesian AH-planes can be embedded in a unique (up to isomorphism) desarguesian PH-plane.

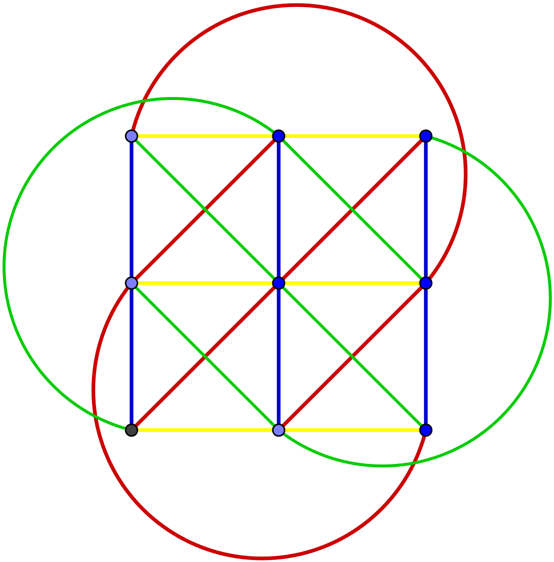
BUT: In general, strongly desarguesian AH-planes can be embedded in a non-desarguesian PH-plane.

Construction

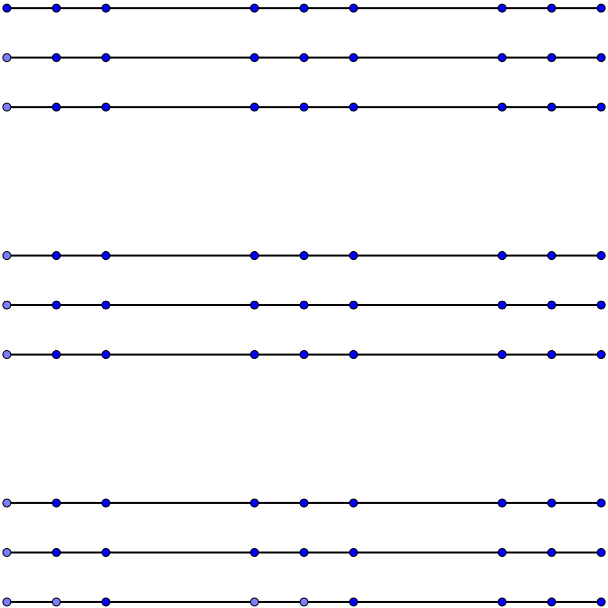
Artmann 1970:

Let $(A, \mathfrak{L})$ be a uniform AH-plane. To embed it, we add new classes of neighbouring points.
For every class of neighbouring points we add an affine plane, which is isomorphic to $(\pi A, \pi \mathfrak{L})$.

affine plane of order 3



uniform AH-plane ,r=3



Remarks

Let $(A, \mathfrak{L})$ be a uniform AH-plane.

The lines consists of parallel classes.

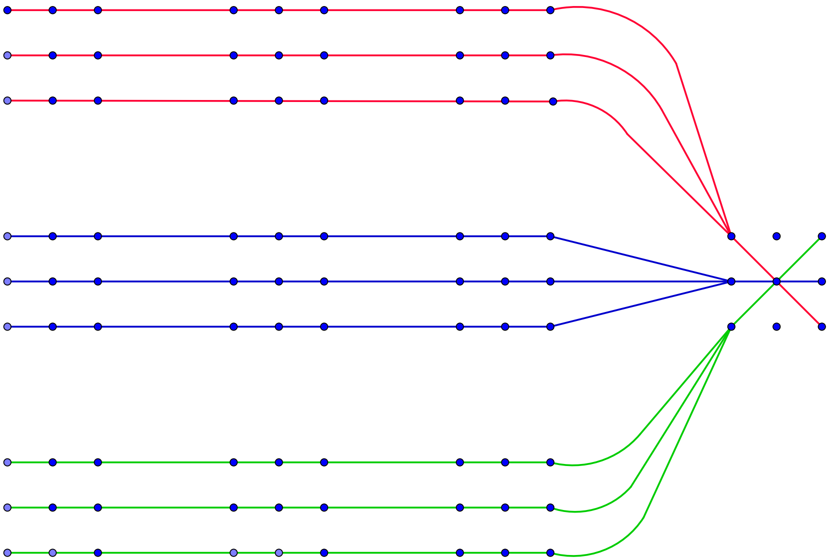
Every parallel class consists of neighbour parallel classes $[[G]] := \{L \in \mathfrak{L} : L \parallel G \text{ and } L \approx G\}$

In every parallel class there are as many neighbour parallel classes as there are lines in one parallel class of $(\pi A, \pi\mathfrak{L})$

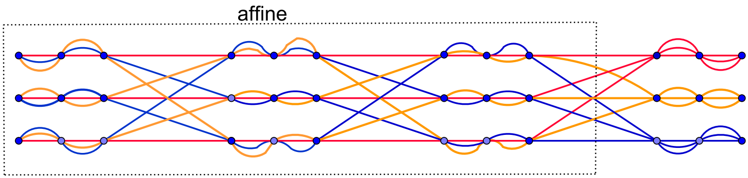
Hence there is a bijection of the neighbour parallel classes on the the parallel classes of $(\pi A, \pi\mathfrak{L})$ -1.

There are as many parallel classes neighbours as there are lines in a parallel class of $(\pi A, \pi\mathfrak{L})$

neighbour parallel classes



one class of neighbour lines



Embedding Theorem

B. Artmann 70, Bacon 74:
Any uniform AH-plane can be embedded into a uniform PH-plane, if it is projectively uniform.

projectively uniform $:\Leftrightarrow$ If $G \approx L$ and $G \cap L = \emptyset$ then $G \parallel L$.

finite planes

Bacon 1974:
Any finite uniform AH-plane is projectively uniform.

W.E. Clark and Drake 1973:
Any finite AH-ring is an H-ring, i.e. (HR3) and (HR4) are in the finite case equivalent.

Corollary: Any finite desarguesian AH-plane is strongly desarguesian.

Section 5

Finite Hjelmslev planes

characteristic numbers

Let $(P, \mathfrak{L})$ be an AH-plane, or PH-plane, respectively and $L \in \mathfrak{L}$, $x \in L$:

r := order of $(\pi P, \pi\mathfrak{L})$
t := number of neighbour points of x on L.

characteristic numbers

Let $(P, \mathfrak{L})$ be an AH-plane, or PH-plane, respectively and $L \in \mathfrak{L}$, $x \in L$:

r := order of $(\pi P, \pi \mathfrak{L})$
t := number of neighbour points of x on L.

t is independent of the choice of x and L.
t is also the number of neighbour lines of L passing x.

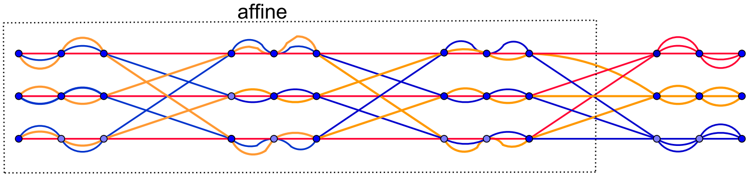
finite PH-plane

Kleinfeld 59, Lüneburg 62:

Let $(P, \mathfrak{L})$ be a finite PH-plane:

- Every point has t^2 neighbour points.
- Every line has t^2 neighbour lines.
- Every point is incident with $t(r+1)$ lines.
- Every line is incident with $t(r+1)$ points.
- $\mid P \mid = t^2(r^2+r+1)$.
- $\mid \mathfrak{L} \mid = t^2(r^2+r+1)$.
- It holds $t = 1$ or $r \leq t$

r=3, t=3



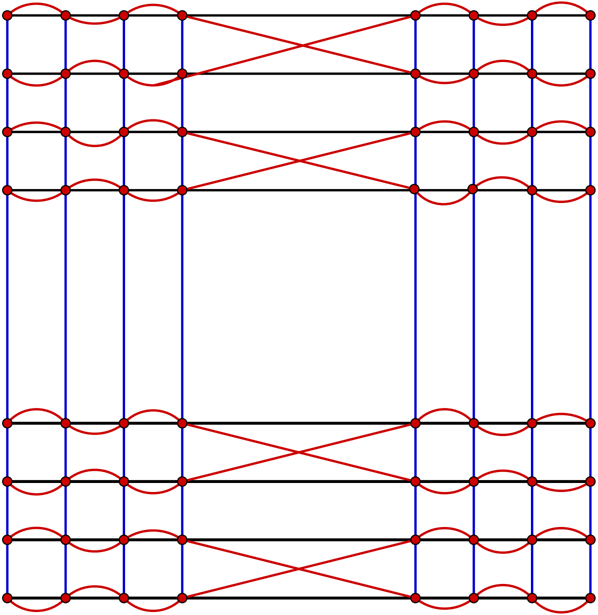
finite AH-plane

Kleinfeld 59, Lüneburg 62:

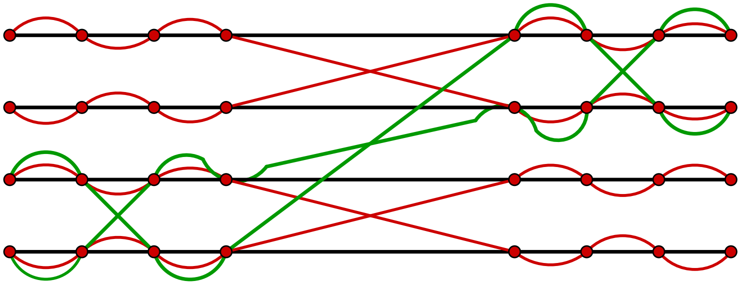
Let $(A, \mathfrak{L})$ be a finite AH-plane:

- Every point has t^2 neighbour points.
- Every line has t^2 neighbour lines.
- Every point is incident with $t(r+1)$ lines.
- Every line is incident with tr points.
- Every line has t^2 parallel lines.
- $\mid A \mid = t^2 r^2$.
- $\mid \mathfrak{L} \mid = t^2(r^2 + r)$.
- It holds $t = 1$ or $r \leq t$

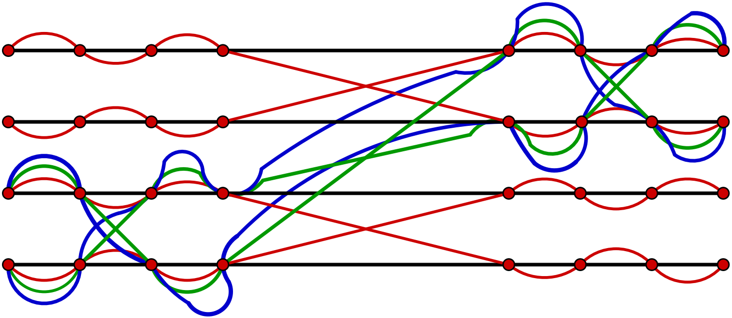
t = 4, r = 2



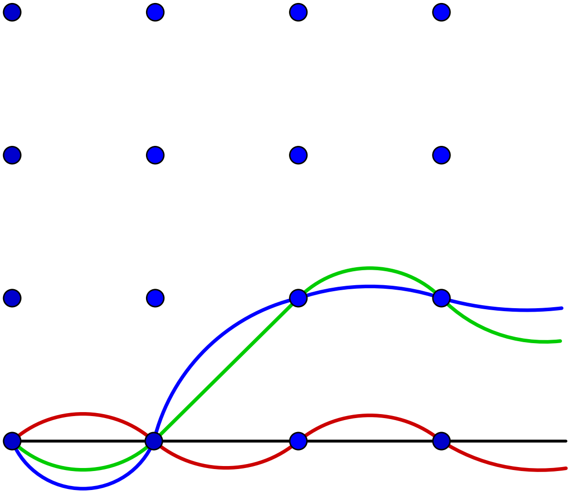
t = 4, r = 2



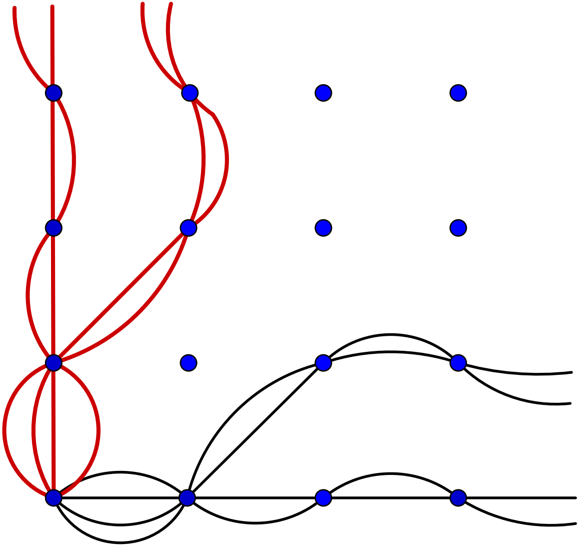
t = 4, r = 2



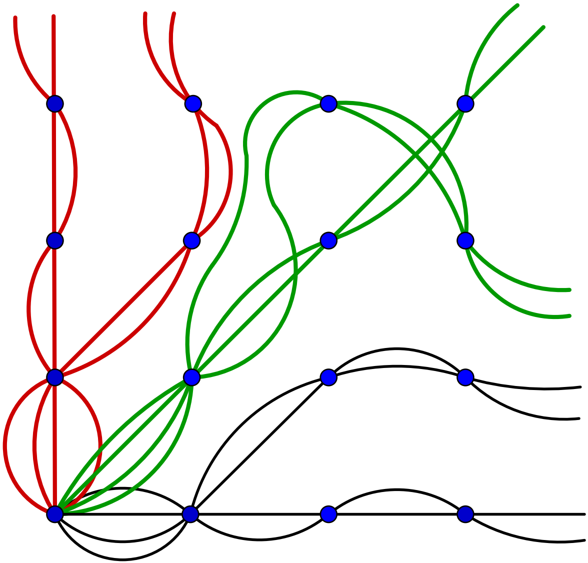
t = 4, r = 2



t = 4, r = 2



t = 4, r = 2



finite and uniform

Let $(P, \mathfrak{L})$ be an AH-plane, or PH-plane, respectively.

Then $(P, \mathfrak{L})$ is uniform, if and only if $r = t$.

finite and commutative

Kleinfeld 59:

There are finite H-rings which are not commutative.

EXAMPLE: Let p be a prime, $K = GF(p^n)$ the finite field with p^n elements, and $\varphi : K \to K$ an automorphism with $\varphi \neq id$ (for example $\varphi(x) = x^p$).

$$R := K \times K, \quad (a, b) + (c, d) := (a + c, b + d), \text{ and}$$
$$(a, b) \cdot (c, d) := (ac, ad + b\varphi(d))$$

Existence-I

R.T. Craig (and later results):
For any affine (projective) plane $(P^*, \mathfrak{L}^*)$ with order r, there is a uniform AH-plane (PH-plane) $P, \mathfrak{L})$ with $(P^*, \mathfrak{L}^*) \cong (\pi P, \pi \mathfrak{L})$.

Existence-II

Artmann 71: If there is an affine plane with order r, then there exists a AH-plane with the characteristic pair $(t, r) = (r^n, r)$

Existence-III

Until 1975 for all known finite AH-planes, t was a power of r.

H.Lenz, Drake 75: Using known results to the number $\ell(t)$ of distinct Latin squares to a given number $t \in \mathbb{N}$, LD constructed new projective Klingenberg planes to a pair (t, r), if:

there is a projective plane with order r
$\ell(t) \geq r + 1$.
Using this results, LD proved:

If there exists a projective plane with order q.

Existence IV

Lenz, Drake 1975:

If there exists a projective plane with order q.
If there exists a PH-plane with pair (t, r) and $q = t(r+1) - 1$.

Then there exists a PH-plane with pair (qt, r) (Lenz pair)

Examples

r	2	2	2	3	3	4	5
t	2	4	8	3	27	4	5
q	5	11	23	11	107	19	29
qt	10	44	184	33	2889	76	145

(There are infinite many possibilities)

Last remarks

Drake and Törner 76,77 give a characterization of PH-planes with Lenz pair (regular, minimal uniform).

Drake, E.E. Shult give examples for PH-planes, in particular distinct PH-planes for $(t, r) = (4, 2)$ or $(8, 2)$

G. Törner considered nearly affine Hjelmselv planes

Lenz, 78, D. Jungnickel 77,78,79,80, S.S. Sane 80, 81, Drake 80 considered finite Hjelmslev and Klingenberg planes.

Open problems

Exists a PH-plane to the pair (6,2) ?

There are some results to PH-planes with the pair (8,5) (not regular!).

End

Thank you for your attention!