Calcul et intrication quantiques: représentation unitaire du groupe de Coxeter/Weyl W(E8)

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Groupes, géométrie discrète et information quantique
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Abstract

- Les paradoxes quantiques d'après Mermin: matrice tressée et matrice CPT,
- Les mesures de l'intrication: tangles (états GHZ, W et CPT),
- Groupes de Pauli, de Clifford et de Coxeter/Weyl,
- Tout est intriqué dans $W(E_8)$:

$$\text{matrice } CPT \Rightarrow_{Swap} \mathcal{C}_3^+ \Rightarrow_{Tof} W(E_8)$$

Exchange matrices 1

- **Qubits** $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_x |0\rangle = |1\rangle, \quad \sigma_x |1\rangle = |0\rangle.$$

- **Two-qubits and the *Swap* gate**

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |0\rangle, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |1\rangle \ldots$$

$$Swap = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad Swap |01\rangle = |10\rangle, \quad Swap |10\rangle = |01\rangle.$$

Exchange matrices 2 [1]

- **Two-qubits and the *Cnot* gate**

$$Cnot = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Cnot\,|10\rangle = |11\rangle\,, \quad Cnot\,|11\rangle = |10\rangle\,.$$

Generation of **entangled states**:
$Cnot(\alpha\,|0\rangle + \beta\,|1\rangle)\,|0\rangle = a\,|00\rangle + b\,|11\rangle$,

- **Three-qubits and the Toffoli gate.**

$$Tof = C^2not = \begin{pmatrix} 1 & 0 \\ 0 & Cnot \end{pmatrix}, \quad Tof\,|110\rangle = |111\rangle\,, \quad Tof\,|111\rangle = |110\rangle\,.$$

[1] Nielsen & Chuang *Quant. Comp. and Quant. Inf.* Cambridge Press (2000)

Mermin square[2]

- Pauli matrices
  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Mermin's square and the real T

$$\begin{matrix} \sigma_x \otimes \sigma_x & \sigma_y \otimes \sigma_y & \sigma_z \otimes \sigma_z \\ \sigma_y \otimes \sigma_z & \sigma_x \otimes \sigma_z & \sigma_x \otimes \sigma_y \\ \sigma_z \otimes \sigma_y & \sigma_z \otimes \sigma_x & \sigma_y \otimes \sigma_x \end{matrix}$$

- Four-dim Kochen Specker theorem

$$\prod \text{eigenvalues} = 1 \quad \text{and} \quad \prod \text{observables} = -1$$

[2] Mermin N D 1993 *Rev. Mod. Phys.* **65** 803,
Planat M and Saniga M 2008 *Quant. Inf. Comp.* **8** 127.

Two Mermin's real triples [3]

Two Mermin's triples of (mutually commuting and real)

$$\{\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z\} \text{ and } \{\sigma_x \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_y \otimes \sigma_y\}.$$

- Joined eigenstates of the first triple as the rows of a orthogonal matrix (braiding matrix)

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} + & + & - \\ - & - & - \\ - & + & + \\ + & - & + \end{pmatrix}.$$

- Joined eigenstates of the second triple as the rows of the entangling orthogonal matrix (CPT matrix)

$$S = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \\ + & + & + \end{pmatrix}.$$

[3] Planat M 2009 Preprint 0904.3691 (quant-ph).

Octahedral symmetry

$$RS = H \otimes I \text{ with } H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$G_{96} = \langle R, S \rangle \cong \mathcal{U}_{13} \cong \mathbb{Z}_4.\mathbb{S}_4,$$

Smallest degree invariant of $\mathcal{U}_{13}$:

$$\mathcal{W} := \alpha^8 + 14\alpha^4\beta^4 + \beta^8.$$

Smallest degree invariant of G_{96}:

$$W^{(2)} := \Sigma_8 + 14\Sigma_{4,4} + 168\Sigma_{2,2,2,2},$$

in the notations of [4], i.e. $\Sigma_8 = \sum_{i=1}^4 \alpha_i^8$, $\Sigma_{4,4} = \sum_{j>i} \alpha_i^4\alpha_j^4$ and $\Sigma_{2,2,2,2} = \prod_{i=1}^4 \alpha_i^2$.

[4] Nebe G, Rains E M and Sloane N J A 2001 *Designs, Codes and Cryptography* **24** 99.

CPT group $\cong \mathcal{P}$, Dirac group $\cong \mathcal{P}_2$ [5]

- Generators of the CPT group are

$$P = i\gamma_0 \quad C = i\gamma_2\gamma_0 \quad \text{and} \quad T = \gamma_3\gamma_1,$$

where the gamma matrices involved are $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$ ($k = x$, y and z), with σ_x, σ_y and σ_z the Pauli spin matrices.

$$CPT \text{ group} \cong \mathcal{P} = \langle \sigma_x, \sigma_y, \sigma_z \rangle \equiv G(4,2,2) \cong Q \rtimes \mathbb{Z}_2,$$

- With the chirality matrix $\gamma_5 = \sigma_x \otimes 1$,
the Dirac group $\cong \mathcal{P}_2$ (the 2-qubit Pauli group) is generated.

[5] Socolovsky M 2004 *Int. J. Theor. Phys.* **43** 1941.

Concurrence [6]

- **Concurrence** $C(\psi) = |\langle\psi|\tilde{\psi}\rangle|$ between the original and flipped state $\tilde{\psi} = \sigma_y |\psi^*\rangle$. Spin-flipped density matrix

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),$$

  As both ρ and $\tilde{\rho}$ are positive, $\rho\tilde{\rho}$ also has only *real and non-negative eigenvalues* λ_i

$$C(\rho) = \tau^{1/2} = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}.$$

- For a two-qubit state $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$, $C = 2|\alpha\delta - \beta\gamma|$, and thus satisfies the relation $0 \leq C \leq 1$, with $C = 0$ for a *separable state* and $C = 1$ for a *maximally entangled state*.

[6] W. K. Wootters, PRL 80, 2245 (1998)

Three-tangle of a three-qubit state [7]

$$|\psi\rangle = \sum_{a,b,c=0,1} \psi_{abc}\,|abc\rangle\,,$$

SLOCC invariant three-tangle: $\tau^{(3)} = 4\,|d_1 - 2d_2 + 4d_3|\,,$

$$d_1 = \psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{100}^2\psi_{011}^2,$$

$$d_2 = \psi_{000}\psi_{111}(\psi_{011}\psi_{100} + \psi_{101}\psi_{010} + \psi_{110}\psi_{001})$$
$$+\psi_{011}\psi_{100}(\psi_{101}\psi_{010} + \psi_{110}\psi_{001}) + \psi_{101}\psi_{010}\psi_{110}\psi_{001},$$
$$d_3 = \psi_{000}\psi_{110}\psi_{101}\psi_{011} + \psi_{111}\psi_{001}\psi_{010}\psi_{100}.$$

- **For the GHZ state** $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$: $\tau^{(3)} = 1$, $\tau = 0$ and $\tau^{(3)} = 0$ for a factorized state. **For a state of the W-class** $|\psi\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, $\tau^{(3)} = 0$, $\tau = \frac{4}{9}$. The 3-tangle is a *residual tangle*
$$\tau^{(3)} = \tau_{A(BC)} - (\tau_{AB} + \tau_{AC}),$$

[7] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev. A* **61** 052306.

$$|CPT\rangle = \frac{1}{2}(|000\rangle + |101\rangle + |110\rangle + |111\rangle)$$

The three-tangle is $\tau^{(3)} = \frac{1}{4}$ and the density matrices

$$\rho_{BC} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \ \rho_{AB} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}, \ \rho_{AC} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

- Square eigenvalues $\left\{\frac{1}{16}(3+2\sqrt{2}), \frac{1}{16}(3-2\sqrt{2}), 0, 0\right\}$ uniform over the subsystems. **For all CPT states $\tau^{(3)} = \frac{1}{4}$, $\tau_{AB} = \tau_{AC} = \tau_{BC} = \frac{1}{4}$ and the linear entropy is** $\tau_{A(BC)} = \frac{1}{4} + 2\frac{1}{4} = \frac{3}{4}$.
- For a mixed state *(Lohmayer R et al 2006 PRL* **97** *260502.)*

$$|Z\rangle = \sqrt{p}\,|GHZ\rangle - e^{-i\phi}\sqrt{1-p}\,|W\rangle$$

with the same three-tangle $\tau^{(3)} = \frac{1}{4}$, one gets $p \approx 0.70$, the sum of two concurrences $\tau_{AB} + \tau_{AC} \approx 0$ and $\tau_{A(BC)} \approx 0.85$.

- A group W is a **Coxeter group** if it is finitely generated by a subset $S \subset W$ of involutions and pairwise relations

$$W = \left\langle s \in S | (ss')^{m_{ss'}} = 1 \right\rangle, \quad (1)$$

where $m_{ss} = 1$ and $m_{ss'} \in \{2, 3, \ldots\} \cup \{\infty\}$ if $s \neq s'$. The pair (W, S) is a Coxeter system, of rank $|S|$ equal to the number of generators.

- **Symmetries of the** 24**-cell**: the group $W(F_4)$, of order 1152 with diagram $x_1 \; - \; x_2 \; -_4 \; x_3 \; - \; x_4$,
i.e. $x_1^2 = x_2^2 = x_3^2 = x_4^2 = (x_2x_3)^4 = (x_1x_2)^2 = (x_3x_4)^2 = (x_1x_3)^3 = (x_1x_4)^3 = (x_2x_4)^3 = 1$.

Finite Coxeter groups

Type	Group	Order	Rank	Related polytope	Coxeter diagram
A_n	S_{n+1}	$(n+1)!$	n	n-simplex	$x_1 - x_2 \ldots x_{n-1} - x_n$
B_n	$\mathbb{Z}_2^n \rtimes S_n$	$2^n n!$	n	n-hypercube	$x_1 -_4 x_2 \ldots x_{n-1} - x_n$
D_n	$\mathbb{Z}_2^{n-1} \rtimes S_n$	$2^{n-1} n!$	n	demihypercube	$x_1 - - - -x_2$ $x_3 \ldots x_{n-2} - x_{n-1}$
$I_2(p)$	Dih_p	$2p$	2	p-gon	$x_1 -_p x_2$
H_3	**	120	3	icosahedron/dodecahedron	$x_1 -_5 x_2 - x_3$
$\mathbf{F_4}$	**	1152	4	24-cell	$x_1 - x_2 -_4 x_3 - x_4$
G_4	**	1440	4	120-cell/600-cell	$x_1 -_5 x_2 - x_3 - x_4$
E_6	**	51840	6	E_6 polytope	$x_1 - x_2 - x_3 - - -x_4$ $- x_5 - x_6$
E_7	**	2 903 040	7	E_7 polytope	$x_1 - x_2 - x_3 - - -x_4$ $- x_5 - x_6 - x_7$
$\mathbf{E_8}$	**	696 729 600	8	E_8 polytope	$x_1 - x_2 - x_3 - - -x_4$ $- x_5 - x_6 - x_7 - x_8$

Complex reflection groups

- V a complex vector space over $\mathbb{C}$.
- Every reflection $s : V \to V$ of order n over $\mathbb{C}$ satisfies the reflection property

$$s_\alpha(x) = x + (\xi - 1)\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha,$$

  for all $x \in V$, where ξ is *a primitive n-th root of unity*.
- The eigenvector α is such that $s(\alpha) = \xi\alpha$ and (x, y) is a positive definite Hermitian form satisfying $(s(x), s(y)) = (x, y)$.
- **For real reflections** s_α:
  $x \in$ Euclidean space $\mathbb{E}$, $s_\alpha \in O(\mathbb{E})$ and $s_\alpha(x) = x - 2\frac{(\alpha,x)}{(\alpha,\alpha)}\alpha$,

Finite complex reflection groups: classification[8]

- Three infinite families $\{\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}\}$, $\{S_n\}$, $\{G(m,p,n)\}$, and **exceptional cases** $\mathcal{U}_l$, $l = 1..34$. The largest one is $\mathcal{U}_{34} \equiv W(E_8)$.
- The **imprimitive reflection groups**:

$$G(m,p,n) = A(m,p,n) \rtimes S_n,$$

$$A(m,p,n) = \left\{\text{Diag}(\omega_1, \omega_2, \ldots, \omega_{n-1}, \omega_n) | \omega_i^m = 1 \text{ and } (\omega_1 \ldots \omega_n)^{m/p} = 1\right\}$$

$\mathcal{P} \equiv G(4,2,2)$, $\mathcal{C}_1 \equiv \mathcal{U}_9$ (order 192)
$\mathcal{U}_{31}$ subgroup of index 2 in $\mathcal{C}_2$ (order 92160).

- Special cases: $G(1,1,n) := S_n := W(A_{n-1})$, $G(m,m,2) := \text{Dih}_m$, $G(2,2,n) := W(D_n)$.

[8] Shephard G C and Todd J A 1954 *Canadian J Math* **6** 274.

Clifford gates are group operations stabilizing Pauli operations[9]

- Action $g \in \mathcal{P}_n$ on an n-qubit state $|\psi\rangle$ is $g|\psi\rangle$, evolves as $Ug|\psi\rangle$, and can be stabilized by U such that $(UgU^\dagger)U|\psi\rangle = U|\psi\rangle$, with $UgU^\dagger \in \mathcal{P}_n$

$$\mathcal{C}_n = \left\{ U \in U(2^n) | U\mathcal{P}_n U^\dagger = \mathcal{P}_n \right\}.$$

  In view of $U^\dagger = U^{-1}$, any **normal subgroup** $\mathcal{Q}_n = \left\{ UgU^{-1}, g \in \mathcal{Q}_n, \forall U \in \mathcal{C}_n \right\}$ of $\mathcal{C}_n$ may be relevant.

- **Quantum errors** $g \in \mathcal{P}_n$ are similar to **real reflections** s_α (of index α) in $\mathbb{E}$, and the Clifford group action on $\mathcal{P}_n$ is similar to the action of $\mathbb{E}$ in $O(\mathbb{E})$.

$$\forall s_\alpha \in \mathbb{E} \text{ and } t \in O(\mathbb{E}),\ \ t s_\alpha t^{-1} = s_{t_\alpha}.$$

[9] Clark S, Jozsa R and Linden N 2008 *Quantum Inf. Comp.* **8** 106.

Clifford group dipoles $\mathcal{C}_n^{\pm}$

$$\mathcal{C}_n^{\pm} \cong E_{2n+1}^{\pm}.\Omega^{\pm}(2n,2)$$

(extraspecial group $\times$ orthogonal group)

- **Single qubit Clifford group dipoles:**

  $$C_1^{+} = G(4,4,2) \cong D_4, \quad C_1^{-} \cong Q \rtimes \mathbb{Z}_3 \cong SL(2,3).$$

  C_1^{-} is a *optimal* 2-*dim* 2-*design*
- **Two-qubit dipoles:**

  $$\mathcal{C}_2^{+} \cong E_{32}^{+}.\Omega^{+}(4,2) \cong E_{32}^{+} \rtimes S_3^2 \cong W(F_4),$$
  $$\mathcal{C}_2^{-} = \mathcal{B}_2' \cong E_{32}^{-}.\Omega^{-}(4,2) \cong E_{32}^{-}.A_5.$$

  A maximal subgroup of $\mathcal{C}_2^{-} \cong SL(2,5)$ (*a optimal* 2-*dim* 5- *design*)
- **Three-qubit dipoles**

  $$\mathcal{C}_3^{+} \cong E_{128}^{+}.\Omega^{+}(6,2) \cong E_{128}^{+}.A_8,$$
  $$\mathcal{C}_3^{-} = \mathcal{B}_3' \cong E_{128}^{-}.\Omega^{-}(6,2) \cong E_{128}^{-}.W'(E_6).$$

- **The real Clifford group** (of order 2 580 480)

$$\mathcal{C}_3^+ = \langle 1 \otimes S, S \otimes 1, 1 \otimes Swap, Swap \otimes 1 \rangle \, ,$$

is the largest maximal subgroup of $W'(E_8) \cong O^+(8,2)$ ($W(E_8)$of order 696 729 600)

$$\left\langle \mathcal{C}_3^+, Tof \right\rangle \cong W(E_8).$$

- **The three-qubit** CPT **group** so defined reads $\tilde{\mathcal{P}} = \langle K, i, j \rangle$, where $\langle i, j \rangle \cong Q$ and $\langle K, i \rangle \cong D_4$, with CPT-type generators also satisfying

$$\langle K, i, j, Tof \rangle \cong W(E_8).$$

3-qubit CPT group generators

$$i=\frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \end{pmatrix},$$

$$j=\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$K=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

$\langle x, y, Tof\rangle \cong W(E_8)$ with $\mathcal{C}_3^+ = \langle x, y, 1\otimes CZ\rangle$ and $\langle x, y\rangle \cong SL(2,5)$, with GHZ-**type generators**

$$x = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$y = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

$\left\langle \tilde{SL}(2,5), i\right\rangle \cong E_{32}^-.S_5$ (*order* 3840), $\left\langle \tilde{SL}(2,5), j\right\rangle \cong \mathbb{Z}_2.W'(E_6)$ (*order* 51840),

$\left\langle \tilde{SL}(2,5), K\right\rangle \cong W(E_7)$ (*order* 2 903 040).

3-qubit representation $\tilde{\mathcal{P}}_2$ of the Dirac group

A **maximal subgroup of** $\tilde{W}(E_7)$ of order 46080 is $M \cong \mathcal{P}_2.S_6$,

$$\tilde{\mathcal{P}}_2 = \langle g_1, g_2, c_1, c_2, u \rangle ,$$

$$\text{with GHZ type gens } g_1 = \begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix}, \text{ and } g_2 = \begin{pmatrix} R_1 & -R_2 \\ -R_2 & R_1 \end{pmatrix},$$

$$R_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \text{ and } R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

with CPT**-type generators** $2c_1$ **and** $2c_2$

$$\begin{pmatrix} 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix},$$

and the **unentangled generator**

$$u = \begin{pmatrix} -U_1 & 0 \\ 0 & -U_2 \end{pmatrix}, U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } U_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$GHZ\,/\,CPT$ structure of the Dirac group

- The normal (extraspecial) group E_{32}^-:
  by removing GHZ-type gens g_1 or g_2.
  The normal (extraspecial) group E_{32}^+:
  by removing CPT-type gens c_1 or c_2 or the unentangled gen. u
- CPT **group of the Dirac equation** is obtained by removing u from E_{32}^-

$$\langle g_1, c_1, c_2 \rangle \cong \langle g_2, c_1, c_2 \rangle \cong \tilde{\mathcal{P}}_1 \cong \lfloor 16, 13 \rfloor \cong Q \rtimes \mathbb{Z}_2,$$

  By removing u from E_{32}^+ one gets a group isomorphic to the CPT **group of the Dirac field**

$$\langle g_1, g_2, c_1 \rangle \cong \lfloor 16, 12 \rfloor \cong Q \times \mathbb{Z}_2,$$

  or *the false* CPT **group**

$$\langle g_1, g_2, c_1 \rangle \cong \langle g_1, g_2, c_2 \rangle \cong \lfloor 16, 11 \rfloor \cong D_4 \times \mathbb{Z}_2.$$

Socolovsky M 2004 Int. J. Theor. Phys. **43** *1941.*

- An intimate connection between entanglement in quantum computing, finite cristallography, group designs, finite geometries, CPT invariance ...
- The Lie group E_8 is important in many attempts in particle physics [10], and $SL(2,5)$ is used in cosmological context [11].
- Clifford group dipoles even more general [12].

[10] Lisi G 2007 Preprint 0711.0770 [hep-th].
[11] Kramer P 2005 *J. Phys. A: Math. Gen.* **38** 3517.
[12] Planat M 2009 Preprints 0904.3691 and 0.906.1063 (quant-ph).