A ZiF COOPERATION GROUP ON

Finite Projective Ring Geometries: An Intriguing Emerging Link Between Quantum Information Theory, Black-Hole Physics and Chemistry of Coupling

to be held at the
Center for Interdisciplinary Research (ZiF)
Bielefeld University, Bielefeld
Germany

in the period of
August 1 – October 31, 2009

Abstracts of a Series of Lectures to be delivered by the Core Members and Short-Term Invitees

Jordan Systems and Associated Geometric Structures

ANDREA BLUNCK
Department Mathematik, Schwerpunkt Geometrie und Diskrete Mathematik
Universität Hamburg, Bundesstrasse 55
D-20146 Hamburg
Germany
(andrea.blunck@math.uni-hamburg.de)

In this series of lectures we introduce the projective line over a Jordan system. This can be seen as a generalization of the projective line over a ring.

A Jordan system is a substructure of a ring (or an algebra) which is closed with respect to addition (and scalar multiplication) and inversion. Standard examples are special Jordan algebras. A Jordan system can also be seen as a special case of a Jordan pair. We present examples of all these algebraic structures. Moreover, we present, as an associated geometric structure, the projective line over a Jordan system. This projective line naturally appears in the theory of chain geometries: The point set of a chain geometry is the projective line over a K-algebra A (with K a field), and the chains of this geometry are the K-sublines. Each Jordan system J contained in A gives rise to a subgeometry of the chain geometry, with point set the projective line over J. Under certain geometric conditions on the chain geometry, this can be reversed: Each subgeometry can be described with the help of a Jordan system J in A.

As an example we study the geometry of points and circles on a non-degenerate quadric in a projective space over K. This can be described as a subgeometry of the chain geometry over an associated Clifford algebra C by finding a suitable Jordan system J in C.

Moreover, we study morphisms of the algebraic and the geometric structures under consideration.

A series of three lectures to be given on August 21, 22 and 24, 2009.

Dual Polar Spaces and the Geometry of Matrices

HANS HAVLICEK

Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien
Wiedner Hauptstraße 8–10, A-1040 Vienna, Austria
(havlicek@geometrie.tuwien.ac.at)

The study of spaces of real and complex matrices (rectangular, alternating, symmetric, Hermitian) was initiated by L. K. Hua around 1945. His results were extended in various directions, e. g., by considering matrices over an arbitrary commutative or non-commutative fields. Another fruitful generalisation was the investigation of so-called *projective matrix spaces*, which can be viewed as completions of matrix spaces in terms of *elements at infinity*.

The aim of these lectures is to present an overview of ordinary and projective geometry of matrices. In particular, we shall put an emphasis on the following one-one correspondences:

- Projective space of rectangular matrices – Grassmannians
- Projective space of alternating matrices – dual polar spaces formed by maximal singular subspaces of a quadratic form
- Projective space of symmetric matrices – dual polar spaces formed by maximal totally isotropic subspaces of a symplectic form
- Projective space of hermitian matrices – dual polar spaces formed by maximal totally isotropic subspaces of a skew-hermitian form

Particular attention will be paid to the finite case due to its connections with quantum theory.

A series of three lectures to be given on October 12, 13 and 14, 2009.

The Black Hole Analogy (BHA) and Finite Geometry

PÉTER LÉVAY

Department of Theoretical Physics, Institute of Physics
Budapest University of Technology and Economics
H-1521 Budapest, Hungary
(levay@neumann.phy.bme.hu)

Recently striking multiple relations have been discovered between two seemingly unrelated strands of knowledge. One of them is Quantum Information Theory (QIT), and the other is String Theory (ST). The former is dealing with the processing of quantum information in the hope that in the nearby future a realistic implementation of quantum computing will be achieved. The latter is aiming at the ambitious goal of unifying all interactions including gravity in a scheme incorporating the quantum theory of extended objects (strings, membranes, etc). The central idea underlying QIT is to regard quantum entanglement, the "characteristic trait" of quantum theory, as a basic resource. In order to enable a full use of this resource quantum entanglement has to be quantified, i. e. the theory of entanglement measures has to be developed. The central idea of ST is duality, i. e. different consistent quantum theories of extended objects seem to be merely different dual faces of the same (still mysterious) M-theory. In order to achieve a deep understanding of dualities special types of objects exhibiting them are studied: black holes, and their higher dimensional analogues, black strings and black rings. It is interesting to note at this point that the key physical concepts of both QIT and ST are: geometry, information, entanglement and (black hole) entropy.

The basic correspondence between QIT and ST is based on the fact that the mathematical expressions of black hole entropy in some cases are just of the form of multipartite entanglement measures. The duality symmetries exhibited by these entropy formulas are related to the ones of admissible local operations used for entanglement quantification (M. J. Duff, Phys. Rev. D76, 025017 (2007); R. Kallosh and A. Linde, Phys. Rev. D73, 104033 (2006)). Recently, many more such correspondences have been found. The one of utmost importance for the work to be done in Bielefeld is the occurrence of finite geometric structures in the context of the BHA. It was realized that certain black hole solutions in *four* dimensions can be understood as quantum systems containing tripartite entanglement of seven qubits. The underlying geometry of this system is governed by the Fano plane, a well-known object to finite geometers (P. Lévay, Phys. Rev. D75 024024 (2006), S. Ferrara and M. J. Duff, Phys. Rev. D76 025018 (2007).)

Though the Fano plane helped to understand many aspects of this curious type of entangled system, the fine details of the underlying geometry remained obscure. In particular in ST it was realized that instead of the infinite discrete duality group in some cases it is enough to study merely its *finite* subgroups. Hence, the problem was raised to account for such finite subgroups using the techniques of finite geometry. In our recent paper (P. Lévay, M. Saniga and P. Vrana, Phys. Rev. D78, 124022 (2008)), by using the geometry of the so-called split Cayley hexagon of order two, we managed to show that the seven qubit structure is intimately tied to Klein's group of order 168, a finite simple subgroup of the full duality group. This group is a subgroup of the full automorphism group of the hexagon which is in turn a subgroup of the so-called Weyl group — an important finite discrete subgroup of the duality group extensively used in ST. Though we made some progress we did not manage to arrive at a satisfactory picture incorporating all the core features of stringy black hole solutions in *four dimensions.*

Due to these difficulties very recently we have started to concentrate on *five dimensional* black hole solutions. These solutions and their corresponding entangled systems (containing only bipartite entanglement of three qutrits) are simpler. In this case we managed to arrive at a fully satisfactory picture based on generalized quadrangles containing lines featuring three points. We succeeded in describing the Weyl subgroup of the full duality group in terms of the geometry of the unique generalized quadrangle of type $(2, 4)$. The "subgeometries" of this object precisely describe consistent truncations well-known to string theorists. And as an extra bonus we obtained an explicit connection with the theory of Mermin squares. These objects are having an independent relevance within the field of quantum theory. Their existence proves the impossibility of noncontextual hidden variable theories. Miraculously, 120 of these Mermin squares labelled by three-qubit Pauli operators live quite naturally within the structure of the entropy formula (P. Lévay, M. Saniga, P. Vrana and P. Pracna, Phys. Rev. D79, 084036 (2009)).

Having clarified the underlying finite geometry of the black hole entropy in *five* dimensions, one is immediately tempted to generalize these results to understand the *four* dimensional case in the same spirit. This is one of the basic tasks to be achieved during the weeks of this cooperation. The basic unifying agents in this case should be certain geometric hyperplanes of the split Cayley hexagon of order two which were shown to be directly related to this "$4D - 5D$ lift". The study of such hyperplanes will hopefully facilitate a deeper understanding of the physics of the BHA.

A series of three lectures to be given on August 9, 10 and 11, 2009.

Quantum Computing Land with Cristallographic Groups

MICHEL PLANAT
Institut FEMTO-ST, CNRS
32 Avenue de l'Observatoire
F-25044 Besançon Cedex, France
(michel.planat@femto-st.fr)

All the distinctive features of quantum mechanics versus its classical counterpart crucially depend on the non-commutativity of observables. For instance, the existence of disjoint sets of mutually commuting operators implies quantum complementarity, in the sense that precise measurements on one set implies that possible outcomes of measurements on the other set are equally probable. The maximal number of such mutually disjoint sets (also called mutually unbiased bases) and their structure relates on concepts of finite projective geometry (Lévay P, Saniga M and Vrana P 2008 *Phys. Rev. D* **78** 124022) and group theory, that we developed in the past five years (Planat M and Saniga M 2008 *Quant. Inf. Comp.* **8** 127). Among such mutually commuting sets, those sharing a base of entangled states are at the origin of paradoxes named Kochen-Specker or Bell's theorems, that recognize the contradiction between the algebraic structure of eigenvalues/measurements and the corresponding expressions for the eigenstates (Mermin N D 1993 *Rev. Mod. Phys.* **65** 803). We investigated the *zoology* and geometry of all mutually unbiased bases for spaces of prime dimension (Planat M and Rosu H 2005 *Eur. Phys. J. D* **36** 133) or composite dimension (Planat M, Baboin A C and Saniga M 2008 *Int. J. Theor. Phys.* **47** 1127).

Inhabitants in quantum computing world are qubits and tensor products of them, their houses are the mutually commuting sets, finite geometries are the villages and towns. I discovered a large city with 696 729 600 apartments that is, in mathematics, *isomorphic* to the largest crystallographic group (Planat M 2009 Preprint 0904.3691 (quant-ph)), named $W(E_8)$. Groups of three inhabitants/states in the villages are in general highly connected/entangled, their contract is of the type GHZ (true tripartite union), W (three bipartite unions), CPT (a compromise), chain state (a pair can get a divorce) and so on. The population of the Pauli group P_n is 4^{n+1}, meaning that at most n people are connected (by a tensor product). The whole *unitary* world is infinite, but people usually travel in a finite part of it named the Clifford group C_n. The Clifford group divides into males/females and thus into two dipolar groups denoted $C_n^{\pm}$. Transsexual people do exist in C_n, e.g. for $n = 2$, $|P_2| = 64$, $|C_2| = 92160$ and $C_2^+ \cap C_2^- = 8$ (as described in (Planat M and Solé P 2008 *J. Phys. A: Math. Theor.* **42**)). The group C_3^+ is the largest maximal subgroup of $W(E_8)$ and indeed the aforementioned contracts are work contracts.

Following Mermin's great intuition about quantum paradoxes, I have discovered two real two-qubit matrices controlling every action in C_n, the first is a braiding matrix R (of a topological character) and the second is a CPT matrix S (related to charge conjugation C, parity P and time reversal T). The dipolar group of a male type $\mathcal{C}_n^+$ is labelled by S and the dipolar group $\mathcal{C}_n^-$ of female type is labelled by R. The geometry of the octahedron is associated to the group $\langle R, S\rangle$ generated by R and S, as already anticipated in the XIXth century (Klein

F 1956 *Lectures on the icosahedron and the solution of equations of the fifth degree* (Dover: New York)) and thoroughly developed in the context of self-dual codes (Nebe G, Rains E M and Sloane N J A *Self-dual codes and invariant theory* (Berlin: Springer)). By the way, the CPT invariance is expressed, up to an isomorphism, by the single-qubit Pauli group $\mathcal{P}_1$ (Socolovsky M 2004 *Int. J. Theor. Phys.* **43** 1941) and is the kernel of the three-qubit representation of $W(E_8)$, the full Dirac group of gamma matrices is the two-qubit Pauli group $\mathcal{P}_2$ and is generated in $W(E_8)$ by matrices sustaining quantum states both of the GHZ and CPT type. Further type of three-qubit entangled states are carried by larger subgroups such as $W(F_4)$, $W(H_4)$, $W(E_6)$ and $W(E_7)$ yet to be investigated in detail, in relation to the quaternionic and octonionic representations.

All these topics, and related ones, will be introduced and discussed in my series of lectures.

A series of three lectures to be given on August 12, 13 and 14, 2009.

Projective Ring Geometries and Role of Coupling in Molecular Dynamics and Chemistry

PETR PRACNA
J. Heyrovský Institute of Physical Chemistry, v.v.i.
Academy of Sciences of the Czech Republic
Dolejškova 3, CZ-182 23 Prague 8, Czech Republic
(pracna@jh-inst.cas.cz)

The link between finite geometry and dynamics of many-body systems, and their bonding in particular, has evolved from conceptual problems in describing them as open, hierarchic dynamical systems interacting with their environment, in which bonds are not considered as an existing but as an emergent property. In this context, the concept of bond is understood as a kind of constraint of dynamic variables of the system, which interconnects the individual layers of hierarchy. The concept of hierarchy reflects a build-up principle of a more complex system from simpler subsystems, in which these building blocks are regarded consistently again as nested dynamical systems.

As a basic illustrative example can serve the chemical bond between two atoms. Regarded in the above sense, the emergent chemical bond represents a qualitative change of spatial and temporal characteristics of the system. Together with emergence of a new kind of spatial symmetry (change of spherical symmetry of each of the two atoms into a cylindrical symmetry of a composite diatomic molecule) there emerge two new temporal properties corresponding to the periods of vibration and rotation of the molecule. This is accompanied by a collapse of two sets of coordinates for translational motions of the atoms into one of the bound molecule. In addition to this, even this simplest case of chemical bond involves, besides the (in principle hierarchic) separation of nuclear and electronic motions, another constrain which requires that the molecular orbital is occupied by two electrons having opposite spins. The underlying Pauli exclusion principle can be understood as a kind of entanglement having origin in another hierarchic level, which does not carry any characteristic of motion but rather represents certain information coupling. In the world of many-body systems the hierarchic arrangement acquires even more complex structure where dynamical systems are built from nested dynamical subsystems. When such hierarchic systems are analyzed with conventional tools of physical chemistry, there arise severe conceptual difficulties. This is because different kinds of couplings between subsystems and their internal structure, described by the interaction energy, attain extreme values in certain points of their configuration space, some becoming extremely large (diverging) and others conversely becoming extremely small (degenerating to zero). These are exactly the configurations through which the system has to pass when it undergoes what we call a chemical change, i.e. when a bond is created (or broken or rearranged). Both these situations, divergence or degeneration to zero, are connected with some "loss-and-emergence of information" about the symmetry of the system. It is also typical that in the neighborhood of these configurations the molecular system becomes extremely sensitive to interactions with its surroundings.

In such situations the problem of dealing with many degrees of freedom is solved technically by a qualitative change of the description by giving up describing the individual subsystems together with the whole system and by introducing a distribution function for the subsystems. The motion of the system is then understood as an evolution of the distribution function which satisfies a certain "equation of motion." We conjecture that in such a change of description, in which the detailed information about the motion of the system on the microscopic level (change of configuration of the subsystems in time) is lost in some sense, it gets transformed into a different kind of "information coupling," in which the distribution function and the related equation of motion are only part of the representation of the system's dynamics. The complementary part of this information coupling is embodied into commutation properties of operators related to observables of the system. At this point it looks conceptually desirable to employ the finite geometry approach, which inherently contains the relation between the commutation properties of operators and the symmetry properties of the spaces they live in and at the same time the concept of duality of these related spaces. This approach should help to circumvent the divergence problem in describing interactions in complex many-body systems and pass smoothly, without extreme behavior and also loss of information, through these singular points in the configuration space. We have at least the first implication that we can represent a system of two coupled qubits, which can be also regarded as two particles carrying spin one-half, by an object of finite projective geometry (M. Saniga, M. Planat, P. Pracna, Theoretical and Mathematical Physics 155 (2008) 905-913), namely the projective line defined over a ring. It is an objective of the part of the project focused on dynamics of bonding of molecular systems to develop the finite geometry concepts and find their relation to spatial and temporal symmetries of molecular systems undergoing chemical changes. From recent studies of fine structures of finite-dimensional Hilbert spaces (H. Havlicek, M. Saniga, Journal of Physics A: Mathematical and Theoretical, 40 (2007) 943-952; H. Havlicek, M. Saniga, Journal of Physics A: Mathematical and Theoretical, 41 (2008) 015302) we have strong implications that the elements of finite geometries, including both projective and affine structures, are promising tools for incorporating the "information coupling" into physical models of hierarchic dynamical systems. This is because the objects of finite geometries are naturally hierarchic in which the affine (physical) and projective (information) layers "live nested" within each other.

We would like to take the opportunity of bringing together a multidisciplinary research group to initiate discussions about the potential of the build-up principles of finite geometries in addressing the issues of even more complex systems in which the processing of information plays a dominant role over the description in the framework of chemistry and physics. This obviously requires finding of the appropriate representation of the relation between information and (physical) forces on the scale ranging from complex systems, which can be regarded as "living," to the levels of simple chemical and physical systems.

A series of three lectures to be given on October 1, 2 and 3, 2009.

From Pauli Groups to Stringy Black Holes: An Intriguing Finite Geometrical Tour from Projective Ring Lines via Certain Generalized Polygons, their Geometric Hyperplanes and Veldkamp Spaces, to a Set of Remarkable Graphs

METOD SANIGA
Astronomical Institute, Slovak Academy of Sciences
SK-05960 Tatranská Lomnica
Slovak Republic
(msaniga@astro.sk)

We shall deal with a variety of concepts of finite geometry that have recently emerged as an intriguing link between quantum information theory and stringy extremal black holes. The tour will start with the notion of the projective line over a finite ring. Here, a particular focus will be on a free cyclic submodule generated by non-unimodular vectors and on the existence of "outliers," that is, vectors which do not belong to any free cyclic submodules generated by uni-modular pairs. Then, we shall move onto generalized polygons. Here, a particular attention will be paid to the Fano plane, generalized quadrangles with lines of size three, the symplectic generalized quadrangle of order three and two generalized hexagons of order two, as well as to some of their embegdings into projective spaces. Next, we shall introduce the concept of a geometric hyperplane of the point-line incidence geometry in order to show not only how rich the structure of these polygons is, but also how they are related to each other. As a follow-up, the Veldkamp spaces of these remarkable objects will be defined and thoroughly examined, and some interesting number theoretical "coincidences" will be pointed out. Finally, several prominent graphs sitting within our polygons will be highlighted and discussed in detail; these include the famous Heawood graph, the Coxeter graph, the Moebius-Kantor graph and the Pappus graph, each of them being the complement of a distinguished kind of a geometric hyperplane of the two hexagons. A special accent will be put on the mutual relation between various concepts and the way they enter quantum information theory and the physics of certain classes of stringy black holes.

A series of three lectures to be given on August 26, 27 and 28, 2009.

Black Holes, Qubits and Octonions

MICHAEL DUFF
Abdus Salam Professor of Theoretical Physics
Blackett Laboratory, Imperial College London
Prince Consort Road, London SW7 2AZ
United Kingdom
(m.duff@imperial.ac.uk)

We review
1) The black hole/qubit correspondence
2) Octonionic description of seven qubits
3) Freudenthal triple classification of three qubit entanglement, and
4) Wrapped branes as qubits

Lecture to be given on October 6, 2009.

Linear Codes and Geometry over Finite Chain Rings

THOMAS HONOLD

Department of Information and Electronic Engineering
Zhejiang University, 38 Zheda Road
310027 Hangzhou
China
(honold@zju.edu.cn)

This minicourse offers an introduction to linear codes over finite chain rings and the link with projective Hjelmslev geometry. In the second part projective Hjelmslev planes over chain rings of length two — the smallest proper-ring case — will be studied from a classical Galois geometry point-of-view.

Part I: Linear codes and projective Hjelmslev geometry over finite chain rings (modules and linear codes over finite chain rings, introduction to projective Hjelmslev geometry, equivalence of linear codes over finite chain rings and multisets of points in projective Hjelmslev geometries, generalized Gray maps and linearly representable codes).

Part II: Projective Hjelmslev planes over chain rings of length two (Fine structure, generalizations of Singer's Theorem, arcs and blocking sets, hyperovals and ovals).

A couple of lectures to be given on August 17 and 18, 2009.

An Introduction to Geometry over Rings

ALEXANDER KREUZER
Universität Hamburg, Department Mathematik
Bundesstrae 55 (Geomatikum), 20146 Hamburg
Germany
(Alexander.Kreuzer@math.uni-hamburg.de)

In several papers between 1916 an 1942, J. Hjelmslev discussed a "natural geometry" in which some lines may intersect in more than one point. Examples show that geometries over rings realize more than one connecting line for some, but not all points.

Later W. Klingenberg introduced projective and affine *Hjelmslev* planes over local rings H in which for any two elements $a, b \in H$ we have $a \in Hb$ or $b \in Ha$ and $a \in bH$ or $b \in aH$ (*Hjelmslev rings*). Points and lines, respectively, are called *neighbours* if there are two or more connecting lines or intersecting points, respectively. Points which are not neighbours have a single connecting line. The relation "neighbour" is an equivalence relation.

If the restriction of an affine or projective Hjelmslev plane to an equivalence class of neighbouring points is an affine plane, we call it an *uniform* Hjelmslev plane. There are some construction methods for uniform Hjelmslev planes, in particular for the finite case. A generalization of Hjelmslev planes are *Klingenberg planes* in which points exist without a connecting line.

In the talk an introduction to Hjelmslev and Klingenberg planes will be given. The meaning of "Desarguesian" and a construction of uniform Hjelmslev planes will be considered as well as the projective closure of an affine Hjelmslev plane.

Lecture to be given on October 8, 2009.

Galois Fields in Quantum Mechanics

APOSTOL VOURDAS
Department of Computing, University of Bradford
Bradford BD7 1DP
United Kingdom
(a.vourdas@Bradford.ac.uk)

A 'Galois quantum system' in which the position and momentum take values in the Galois field $GF(p^\ell)$, is considered. Displacements in the $GF(p^\ell) \times GF(p^\ell)$ phase space and the corresponding Heisenberg-Weyl group, are studied. Symplectic transformations are shown to form the $Sp(2, GF(p^\ell))$ group.

Frobenius transformations and the corresponding Galois group, are a unique feature of these systems (for $\ell \geq 2$). From a mathematical point of view they introduce algebraic concepts into harmonic analysis. From a physical point of view, if they commute with the Hamiltonian, there are constants of motion which are discussed.

The difference between a Galois quantum system and other finite quantum systems where the position and momentum take values in the ring $[\mathbb{Z}_p]^\ell$, is discussed.

References:

J. Phys. A38, 8453 (2005); J. Math. Phys. 47, 092104 (2006); Acta Appl. Math. 93, 197 (2006); J. Fourier Anal. Appl. 14, 102 (2008)
Reviews: Rep. Prog. Phys. 67 (2004) 267; JPA 40, R285 (2007)

Lecture to be given on August 19, 2009.